MBA semester 1

Quantitative analysis for management decisions

UNIT 1

Introduction to Quantitative Techniques

Definition:

Quantitative Techniques (QT) refer to a set of mathematical, statistical, and scientific methods used to analyze numerical data and make better business decisions.

They help managers solve business problems objectively by using measurable information instead of guesswork.

QT is widely used in planning, forecasting, investment decisions, production management, inventory control, marketing research, and financial analysis.

In simple words:

QT = Using numbers + formulas + data to take better decisions.

1. Planning: Planning means deciding in advance what to do, how to do it, when to do it, and who will do it.

How QT helps in planning

QT provides numerical data, models, and analysis to plan business activities more effectively.

Examples

1.A company uses Linear Programming to decide the optimal combination of products to maximize profit.

2.A bank uses queuing models to plan the number of counters needed to reduce customer waiting time

2. Forecasting: Forecasting means estimating future events like sales, demand, price, or production based on past data.

How QT helps in forecasting

Using methods like time series analysis, regression, and trend analysis, managers can predict future values.

Examples

1.A retail store predicts next month's sales using past 12-month sales data.

2.An airline forecasts future ticket demand based on seasonal trends and previous year traffic.

3. Investment Decisions (Capital Budgeting): Investment decisions involve choosing the best project or asset to invest money in.

How QT helps in investment

QT uses techniques such as:

Net Present Value (NPV)

Internal Rate of Return (IRR)

Payback Period

Risk and return analysis

Examples

1.A company compares two projects using NPV and selects the one with the higher present value.

2.An investor uses probability analysis to choose the best investment portfolio.

4. Production Management: This involves planning and controlling the production process to achieve efficiency.

How QT is used

Linear Programming for optimal resource allocation

Scheduling models for deciding production time

Break-even analysis to determine production level

Examples

1.A factory uses LP to find the best use of raw materials, machinery hours, and labour.

2.A manufacturing unit uses PERT/CPM to schedule production activities to meet deadlines.

5. Inventory Control: Inventory control ensures that the business keeps just the right amount of stock—not too much, not too little.

How QT helps

Using models like:

EOQ (Economic Order Quantity)

Reorder Level (ROL)

Safety Stock calculations

Examples

1.A company calculates EOQ to determine the most economical quantity of goods to order.

2.A pharmacy uses QT to maintain medicines without overstocking or stockouts.

6. Marketing Research: Marketing research involves understanding consumer behavior, product pricing, market demand, etc.

How QT helps

Survey analysis

Market segmentation

Price-demand relationships

Sales forecasting

Examples

1.A company uses regression analysis to study how price changes affect demand.

2.A marketing team conducts a consumer survey and uses statistical tools to analyze preferences.

7. Financial Analysis: Financial analysis studies financial performance, risk, profitability, liquidity, etc.

How QT helps

Ratio analysis

Regression and variance analysis

Portfolio optimization

Trend analysis of financial statements

Examples

1.Using QT, a company analyzes profit trends over 5 years to make future plans.

2.A financial analyst uses correlation to study the relationship between market index and stock prices.

Scope of Quantitative Techniques (QT)

1. Business Forecasting

Forecasting means predicting future values like sales, demand, price, profit, or production using mathematical/statistical techniques.

Explanation

QT uses tools such as time series analysis, moving averages, and regression to study past data and predict future trends.

This helps businesses plan production, staffing, budgeting, and inventory.

Example

A clothing store analyzes the last 2 years of monthly sales using a moving average.

If sales are increasing every festive season, the store forecasts higher demand for October–November and orders more stock.

2. Inventory Management

Inventory management ensures that stock is always available at minimum cost.

Explanation

QT helps determine:

How much to order (EOQ – Economic Order Quantity)

When to order (Reorder Level)

How much safety stock to keep

Example

A company has high ordering cost and carrying cost. Using EOQ formula, it finds that ordering 400 units per order minimizes total cost.

This avoids excess inventory and reduces storage cost.

3. Production Planning & Control: Production planning ensures efficient use of raw materials, machines, and labour.

Explanation

QT helps:

Allocate limited resources

Decide production schedule

Determine optimal output level

Minimize production cost or maximize profit

Techniques used: Linear Programming, PERT/CPM, Scheduling Models

Example

A factory has 100 labour hours and 150 machine hours.

It uses Linear Programming (LPP) to determine how many units of Product A and Product B to produce to maximize profit without exceeding available resources.

4. Financial Management: Financial decisions involve investment, budgeting, cost control, and risk reduction.

Explanation

QT supports:

Capital budgeting (NPV, IRR, Payback)

Risk analysis

Credit scoring

Break-even analysis

Example

A company has two investment projects: Project X and Project Y.

Using Net Present Value (NPV), it finds:

NPV of X = ₹50,000

NPV of Y = ₹32,000

Since X has a higher NPV, the company selects Project X.

5. Marketing Research: Marketing decisions include pricing, product launch, advertising, and customer behavior.

Explanation

QT helps:

Analyze survey data

Study demand and price relationship

Identify market segments

Measure advertising effectiveness

Example

A company uses regression analysis to understand how price affects demand.

It finds:

“If price increases by ₹10, demand decreases by 15 units.”

This helps in setting the right price.

6. Quality Control: Ensuring products meet the required quality standards.

Explanation

Control charts

Sampling techniques

Statistical quality control (SQC)

These help detect defects early and maintain consistency.

Example

1.A biscuit factory checks the weight of 50 packets daily and plots them on a control chart.

2.If the weight goes out of control limits, the manager adjusts the machine to avoid defects.

7. Human Resource Planning: HR planning ensures the right number of employees at the right time.

Explanation

Estimate manpower requirements

Schedule staff

Reduce waiting time

Improve labour efficiency

Techniques: Queuing theory, forecasting, scheduling

Example

A hospital uses queuing models to determine the number of nurses needed during peak hours.

This reduces patient waiting time and improves service quality.

8. Operations Research (OR) & Decision Making: OR uses mathematical modelling to find the best possible solution to business problems.

Explanation

Minimizing cost

Maximizing output

Optimal allocation of resources

Efficient logistics and scheduling

Techniques include:

LPP, Transportation, Assignment, Simulation, Game Theory

Example

A logistics company uses the transportation model to decide how many goods to send from each warehouse to each store so that total transport cost is minimized.

Linear and non-linear functions in quantitative analysis

Quantitative analysis uses mathematical functions to model business situations. Two commonly used types are **Linear** and **Non-linear** functions.

1. Linear functions

Meaning

A **linear function** shows a **constant rate of change** between two variables.
It forms a **straight line** when graphed.

General Form:

$$y = a + bx$$

- **a** = intercept (value of y when x = 0)
- **b** = slope or rate of change (constant)

Characteristics

- Graph is a **straight line**.
- **Slope (b)** is constant.
- Both variables move **proportionately**.
- Suitable for **short-term prediction**.

Business Examples

Example 1: Total Cost Function

$TC=10,000+50Q$

- Fixed cost = ₹10,000
- Variable cost = ₹50 per unit
- For **each extra unit**, cost increases by **₹50** (constant).

Simple Numerical Example

$y=100+5x$

Find y when x = 10:

$$y=100+5(10)$$

$$y=100+50$$

$$y=150$$

2. Non-linear functions

Meaning

A **non-linear function** does **not change at a constant rate**.
Graph is **curved**, not a straight line.

The relationship between variables may be:

- Increasing at an increasing rate
- Increasing at a decreasing rate
- Decreasing at a non-constant rate

General Forms

Includes many types:

1. Quadratic (Parabola) $y=a+bx+cx^2$
2. Exponential $y=ab^x$
3. Logarithmic $y=a+b\ln x$
4. Power Function $y=ax^b$

Characteristics

- Rate of change is **not constant**.
- Graph is **curved**.
- Used when variables grow or decline **non-proportionately**.

Difference Between Linear and Non-Linear Functions:

basis	linear function	non-linear function
Graph	Straight line	Curved line
rate of change	Constant	Not constant
formula	y=a+bx	Many types (quadratic, exponential, etc.)
use in basis	Cost, revenue, forecasting	Growth, learning curve, diminishing returns
complexity	Simple	More complex

Business Applications of linear and non-linear functions

<u>Linear function</u>

A linear function shows a **constant relationship** between two business variables. This means when one variable increases, the other increases (or decreases) **at a fixed, predictable rate**.
Because of this, linear functions are easy for businesses to use in short-term planning.

Business Applications of Linear Functions

1. **Cost estimation:**
   Businesses use linear functions to estimate total cost because variable cost per unit usually remains constant.
2. **Sales and revenue forecasting:**
   When price is constant, revenue increases proportionally with the number of units sold.
3. **Profit planning:**
   Profit changes steadily as sales or production increases.
4. **Budgeting and short-term forecasting:**
   Since the change is predictable, linear relationships help businesses plan short-term financial goals.

Non-Linear Functions

A non-linear function shows a **changing relationship** between two variables.
The rate of change is not fixed; it may increase or decrease unevenly.
These functions describe more complex business situations.

Business Applications of Non-Linear Functions

1. **Diminishing returns in production:**
   When a company adds more labour or machinery, output increases at first but eventually starts increasing at a slower rate.
2. **Demand curve behaviour:**
   When price changes, customer demand usually does not change in a straight line. Sometimes demand drops slowly, and sometimes it drops sharply.
3. **Business growth models:**
   Company profits, market size, and population usually grow at increasing or decreasing rates, not in a straight line.
4. **Learning and efficiency:**
   As workers gain experience, the time taken to produce a product decreases in a non-linear pattern.

Limits of functions

Meaning

The limit of a function describes the **expected value** of something when the input moves closer and closer to a particular point

Business Applications

1. **Marginal analysis:**
   Limits help in understanding how cost or revenue changes when production changes by a very small amount.
2. **Forecasting:**
   Limits are used to understand how sales, costs, or prices behave as they approach certain conditions or boundaries.
3. **Optimisation:**
   Economists use limits to find maximum profit, minimum cost, and optimal production levels.
4. **Smooth behaviour analysis:**
   Limits help to check if a function behaves steadily near a particular point.

Continuous Functions

Meaning

A continuous function is one that has **no breaks, gaps, or jumps** in its graph.
It behaves smoothly.
You can draw the entire graph without lifting the pencil.

This means the value of the function changes gradually and predictably.

Business Uses

1. **Smooth cost behaviour:**
   Businesses assume costs change smoothly with changes in production, making analysis more accurate.
2. **Continuous demand patterns:**
   Small changes in price usually cause small, smooth changes in demand.
3. **Revenue and profit forecasting:**
   Continuous functions help create more realistic financial projections.
4. **Decision-making models:**
   Optimization in business (maximizing profits, minimizing costs) usually requires continuous functions because they behave predictably.

properties of limits :

* $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
* $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
* $\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
* $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$
* $\lim_{x \to a} \sin f(x) = \sin\left[\lim_{x \to a} f(x)\right]$
* $\lim_{x \to a} \log f(x) = \log\left[\lim_{x \to a} f(x)\right]$
* $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$
* $\lim_{x \to a} \frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$
* $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$
* $\lim_{x \to a} \frac{e^{x-1}}{x} = 1$
* $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = 1$
* $\lim_{n \to \alpha} \left[1 + \frac{1}{n}\right]^n = 0$
* $\lim_{x \to 0} \frac{a^x - 1}{x} = \log a$

Problems:-

1. Evaluate $\lim_{x\to 2}$ the given function $\frac{x^2-5x+6}{x^2+x-6}$

Sol $\lim_{x\to 2} \frac{x^2-5x+6}{x^2+x-6}$

let $(x=2)$

$$\Rightarrow \frac{x^2-5x+6}{x^2+x-6}$$

$$\Rightarrow \frac{2^2-5(2)+6}{2^2+2-6}$$

$$\Rightarrow \frac{4-10+6}{4+2-6}$$

$$= \frac{-6+6}{4-4}$$

$$= \frac{0}{0}$$

$$= 0$$

It is indetermination form so given function

$$\lim_{x\to 2} \frac{x^2-5x+6}{x^2+x-6}$$

Now, find the roots of numerator & denominator.

$$x^2-5x+6. \qquad 6x^2 \quad -2x \quad -3x \qquad (\text{or}) \quad \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$x^2-2x-3x+6$$

$$(x^2-2x)-3(x-2)$$

$$x(x-2)-3(x-2)$$

$(x-2)(x-3)=0.$

Now in denominator function

$f(x) = x^2+x-6$

$= x^2-2x+3x-6$

$= x(x-2)+3(x-2)$

$= (x-2)(x+3)$

$-6x^2$

$-2x \quad 3x.$

sub the numerator & denominator roots

$$\lim_{x\to 2} \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+3)}$$

$$\lim_{x\to 2} = \frac{(x-3)}{x+3}$$

$$= \frac{2-3}{2+3}$$

$$= \frac{-1}{5}$$

$$\lim_{x\to 2} f(x) = \frac{-1}{5}$$

2) $\lim_{x\to 2} \frac{x^2+5x+6}{2x^2-3x}$

Sol $\lim_{x\to 2} \frac{x^2+5x+6}{2x^2-3x}$

let $(x=2)$

$$\frac{2^2+5(2)+6}{2(2)^2-3(2)}$$

$$= \frac{4+10+6}{8-6}$$

$$= \frac{10+10}{2}$$

$$= \frac{20}{2}$$

$$\underset{x \to 2}{lt} f(x) = 10$$

3) $\underset{x \to 3}{lt} \quad x+2$

$$\underset{x \to 3}{lt} \quad x+2$$

let $x = 3$

$$\underset{x \to 3}{lt} = x+2$$

$$= 3+2$$

$$\underset{x \to 3}{lt} f(x) = 5$$

4) $\underset{x \to 4}{lt}$ function $\frac{x^2-16}{x-4}$

Sol $\underset{x \to 4}{lt} \quad \frac{x^2-16}{x-4}$

$$\Rightarrow \frac{x^2-4^2}{x-4} \qquad (\because x^2-4^2 = a^2-b^2 = (a+b)(a-b))$$

$$\Rightarrow \frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}}$$

$$\underset{x \to 4}{lt} \Rightarrow (x+4)$$

let $x=4$

$\lim_{x\to 4} (x+4)$

$4+4=8$

$\lim_{x\to 4} f(x)=8$

5) $\lim_{x\to 2} \frac{(x+4)(2x+1)}{x^2+5x+2}$

Sol $\lim_{x\to 2} \frac{(x+4)(2x+1)}{x^2+5x+2}$

let $x=2$

$= \frac{(2+4)(2(2)+1)}{2^2+5(2)+2}$

$= \frac{(6)(5)}{4+10+2}$

$= \frac{30}{16}$

$= \frac{15}{8}$

$\lim_{x\to 2} f(x) = \frac{15}{8}$

6) $\lim_{x\to 1} x^2+2x+3$

$\lim_{x\to 1} x^2+2x+3$

let $x=1$

$= 1^2+2(1)+3$

$= 1+2+3$

$\lim_{x\to 1} = 6$

7) $\lim_{x \to 0} \frac{1}{x^2-3x+2}$

$\lim_{x \to 0} \frac{1}{x^2-3x+2}$

Let $x=0$

$= \frac{1}{0^2-3(0)+2}$

$= \frac{1}{2}$

$\lim_{x \to 1} f(x) = \frac{1}{2}$

8) $\lim_{x \to \alpha} \frac{3x^2-5x+9}{2x^2+4x+7}$

Sol $\lim_{x \to \alpha} \frac{3x^2-5x+9}{2x^2+4x+7}$

$= \frac{3(\alpha)^2-5(\alpha)+9}{2(\alpha)^2+4(\alpha)+7}$

It undefine form Now convert α too

$x \to \alpha$

$x = \alpha$

$\frac{1}{x} = 0$

Divide the entire function with x^2

$$\lim_{\frac{1}{x} \to 0} \frac{\frac{3x^2-5x+9}{x^2}}{\frac{2x^2+4x+7}{x^2}}$$

$$\lim_{\frac{1}{x}\to 0} \frac{\frac{3x^2}{x^2} - \frac{5x}{x^2} + \frac{9}{x^2}}{\frac{2x^2}{x^2} + \frac{4x}{x^2} + \frac{7}{x^2}}$$

$$= \frac{3 - \frac{5}{x} + \frac{9}{x^2}}{2 + \frac{4}{x} + \frac{7}{x^2}}$$

$$= \frac{3 - 5\left(\frac{1}{x}\right) + 9\left(\frac{1}{x^2}\right)}{2 + 4\left(\frac{1}{x}\right) + 7\left(\frac{1}{x^2}\right)}$$

$$= \frac{3 - 5(0) + 9\left(\frac{1}{0}\right)}{2 + 4(0) + 7\left(\frac{1}{0}\right)}$$

$$= \frac{3}{2}$$

$$\lim_{\frac{1}{x}\to 0} = \frac{3}{2}$$

9) $\lim_{x\to \infty} \frac{\sqrt{9x^2+5x-7} - \sqrt{4x^2+7x+5}}{2x^2-3}$

Sol $\lim_{x\to \infty} \frac{\sqrt{9x^2+5x-7} - \sqrt{4x^2+7x+5}}{2x-3}$

It is undefine form so convert ∞ too

$x \to \infty$

$x = \infty$

$\frac{1}{x} = 0$

Divide the Entire function with x^2

$$\lim_{\frac{1}{x} \to 0} \frac{\sqrt{\frac{9x^2}{x^2} + \frac{5x}{x^2} - \frac{7}{x^2}} - \sqrt{\frac{4x^2}{x^2} + \frac{7x}{x^2} + \frac{5}{x^2}}}{\frac{2x^2}{x^2} - \frac{3}{x^2}}$$

$$= \frac{\sqrt{9 + \frac{5}{x} - \frac{7}{x^2}} - \sqrt{4 + \frac{7}{x} + \frac{5}{x^2}}}{2 - \frac{3}{x^2}}$$

$$= \frac{\sqrt{9+5\left(\frac{1}{x}\right) - 7\left(\frac{1}{x^2}\right)} - \sqrt{4 + 7\left(\frac{1}{x}\right) + 5\left(\frac{1}{x^2}\right)}}{2 - 3\left(\frac{1}{x^2}\right)}$$

$$= \frac{\sqrt{9+5(0) - 7\left(\frac{1}{0}\right)} - \sqrt{4+7(0)+5(0)^2}}{2 - 3\left(\frac{1}{0}\right)}$$

$$\lim_{x \to 0} = \frac{\sqrt{9} - \sqrt{4}}{2}$$

$$= \lim_{x \to 0} \frac{3-2}{2}$$

$$= \lim_{x \to 0} = \frac{1}{2}$$

Polynomial function :- polynomial function is a function of the form $y = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots a_n x^{n-1}$

Eg :- * $x^4 + 2x^3 + 3x^2 + x + 1$

* $x^2 + 2x + 1$

Rationalization function :- A function defined by quality of two polynomials is called Rationalization function. Ex:- $3x+4$.

$$y = \frac{x^2+5x+6}{2x^2-3x}$$

$$y = \frac{x^2-5x+6}{x^2+x-6}$$

Irrationalization function :- A function is of the form based on the roots is called a irrationalized function.

Ex :- $y = \sqrt{x^2+1}$

1. $\lim_{x \to 0} \sqrt{5+x} - \sqrt{5-x} / x$

$$\lim_{x \to 0} \frac{\sqrt{5+x} - \sqrt{5-x}}{x}$$

let $x = 0$

$\lim_{x \to 0}$

$$\lim_{x \to 0} \frac{\sqrt{5+0} - \sqrt{5-0}}{0}$$

$$= \frac{\sqrt{5} - \sqrt{5}}{0}$$

$$= 0$$

It is undefined function

Rationalizatton the given function

$$\lim_{x \to 0} \frac{\sqrt{5+x} - \sqrt{5-x}}{2} \times \frac{\sqrt{5+x} + \sqrt{5-x}}{\sqrt{5+x} + \sqrt{5-x}}$$

$$\lim_{x \to 0} \frac{(\sqrt{5+x})^2 - (\sqrt{5-x})^2}{x(\sqrt{5+x} + \sqrt{5-x})]}$$

$$\lim_{x \to 0} \frac{\cancel{5} + x - \cancel{5} + x}{2(\sqrt{5+x} + \sqrt{5-x})}$$

$$\lim_{x \to 0} \frac{2\cancel{x}}{\cancel{x}(\sqrt{5+x} + \sqrt{5-x})}$$

$$\lim_{x \to 0} \frac{2}{\sqrt{5+x} + \sqrt{5-x}}$$

$$\lim_{x \to 0} \frac{2}{\sqrt{5+x}} \quad \lim_{x \to 0} \frac{2}{\sqrt{5-x}}$$

$$= \frac{2}{\sqrt{5+0}} + \frac{2}{\sqrt{5-0}}$$

$$= \frac{2}{\sqrt{5}} + \frac{2}{\sqrt{5}}$$

$$= \frac{4}{\sqrt{5}}$$

$$\lim_{x \to 0} f(x) = \frac{4}{\sqrt{5}}$$

2) $\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x}$

Sol) let

$\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x}$

let $x = 0$

$\frac{\sqrt{a+0} - \sqrt{a-0}}{0}$

It is undefined form.

Rattonalizatton the given function.

$\lim_{x \to 0} \frac{\sqrt{a+x} - \sqrt{a-x}}{x} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}}$

$\lim_{x \to 0} \frac{(\sqrt{a+x})^2 - (\sqrt{a-x})^2}{x[\sqrt{a+x} + \sqrt{a-x}]}$

$\lim_{x \to 0} \left[\frac{a+x-(a-x)}{x(\sqrt{a+x} + \sqrt{a-x}}\right]$

$\lim_{x \to 0} \frac{a+x-a+x}{x(\sqrt{a+x} + \sqrt{a-x})}$

$\lim_{x \to 0} \frac{2}{\sqrt{a+x} + \sqrt{a-x}}$

$\Rightarrow \frac{2}{\sqrt{a+0}} + \frac{2}{\sqrt{a-0}}$

$\Rightarrow \frac{2}{\sqrt{a}} + \frac{2}{\sqrt{a}}$

$\lim_{x \to 0} f(x) \Rightarrow \frac{4}{\sqrt{a}}$

3) $$\lim_{x \to 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$$

Sol Let

$$\lim_{x \to 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}}$$

Let $x=0 \Rightarrow \frac{0}{\sqrt{1+0} - \sqrt{1-0}}$

$$= \frac{0}{\sqrt{1} - \sqrt{1}}$$

$$= \frac{0}{0}$$

It is undefined function.

Rationalization the given function.

$$\lim_{x \to 0} \frac{x}{\sqrt{1+x} - \sqrt{1-x}} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$\lim_{x \to 0} \frac{x(\sqrt{1+x}) + \sqrt{1-x})}{(\sqrt{1+x})^{\cancel{2}} - (\sqrt{1-x})^{\cancel{2}}}$$

$$\lim_{x \to 0} \frac{x(\sqrt{1+x} + \sqrt{1-x})}{\cancel{1}+x-\cancel{1}+x}$$

$$\lim_{x \to 0} \frac{\cancel{x}(\sqrt{1+x} + \sqrt{1-x})}{2\cancel{x}}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x} + \sqrt{1-x}}{2}$$

$$\lim_{x \to 0} \frac{\sqrt{1+x}}{2} + \lim_{x \to 0} \frac{\sqrt{1-x}}{2}$$

$$= \frac{\sqrt{1+0}}{2} + \frac{\sqrt{1-0}}{2}$$

$$= \frac{\sqrt{1}}{2} + \frac{\sqrt{1}}{2} = \frac{\cancel{2}\sqrt{1}}{\cancel{2}} = \sqrt{1} = 1.$$

$$\therefore \lim_{x \to 0} = 1$$

Homework. $\lt_{x \to 1} \dfrac{x^2-3x+2}{5x^2-4x-1}$

Continuous function :- A continuous function which satisfy the RHS which equals to the LHS then it is said to be a continuous function.

1) $f(x) \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x \ge 1 \end{cases}$

Sol G.T $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x \ge 1 \end{cases}$

to limit range is 1

L.H.S	R.H.S
$\lt_{x \to 1} f(x)$	$\lt f(x)$
$\lt_{x \to 1} x^2$	$\lt_{x \to 1} x$
$(1)^2$	(1)
$=1$	$=1$

LHS = RHS.

2) $f(x) = \begin{cases} \frac{1}{2}(x^2-4) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2-8x^{-3} & \text{if } x > 2 \end{cases}$

Sol The limit Range is 2

L.H.S

If $x \to 2$ $\lim_{x\to 2} f(x) = \frac{1}{2}(2^2-4)$

$= \frac{1}{2}(4-4)$

$= \frac{1}{2}(0)$

$= 0.$

R.H.S

$\lim_{x\to 2} f(x)$ $\lim_{x\to 2} 2-8x^{-3}$

$\lim_{x\to 2} 2-8x^{1/3}$

$\lim_{x\to 2} 2-8(2)^{-1/3}$

$2-2^3(2)^{1/3}$

$= 2-2$

$\lim_{x\to 2} = 0$

LHS = RHS.

3) Homework.

Check it is a continuous function.

$$f(x) \begin{cases} \frac{x}{2}, & \text{if } x<2 \\ x^{2/3} & \text{if } x>2 \end{cases}$$

4) let $f(x) = \begin{cases} x^2, & 1 \le x \le 2 \\ x-3 & x \ge 2 \end{cases}$ find the given function

Continuous or not

5) $f(x) = \begin{cases} x+2 & -1 < x < 3 \\ x^2 & 3 < x < 5 \end{cases}$

6) $f(x) = \begin{cases} x+1 & \text{if } x<1 \\ 2x & \text{if } 1<x<2 \\ 1+x^2 & \text{if } x \geq 2 \end{cases}$

Sol The limit Range is 1, 2

The limit Range is 1

LHS $\lim_{x\to 1} x+1$

$1+1$

$=2$

RHS $\lim_{x\to 1} 2x$

$2(1)$

$=2$

L.H.S = R.H.S

$\lim_{x\to 1}$ It is a continuous function

The limit range is "2"

L.H.S $\lim_{x\to 2} 2(x)$

$=2(2)$

$=4$

R.H.S $\lim_{x\to 2} 1+x^2$

$1+(2)^2$

$1+4$

$=5$

L.H.S $\neq$ R.H.S

$\lim_{x\to 2}$ It is not a continuous function.

7) practice $f(x) = \begin{cases} 4-x^2 & \text{if } x\leq 0 \\ x-5 & \text{if } 0<x\leq 1 \\ 4x^2-9 & \text{if } 1<x<2 \\ 3x+4 & \text{if } x\geq 2 \end{cases}$

UNIT- II

Differentiation and Integration.

Differentiation :- Differentiation is the process of finding the rate of change of a function at any given point. It Means how are Quantity change in relation to another. The result of this process derivative.

Formula :- The given function $f(x)$ is said to be differentiative function then we need to find $f'(x)$

$$\therefore f'(x) = \frac{d}{dx}[f(x)]$$

Basic differentiation formulas :-

1) $\frac{d}{dx}(x) = 1$

2) $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$

3) $\frac{d}{dx}(a) = 0$ ($\because$ where a is constant).

4) $\frac{d}{dx}(ax) = a$

5) $\frac{d}{dx}(\log x) = \frac{1}{x}$

6) $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$

7) $\frac{d}{dx}(e^x) = e^x$

8) $\frac{d}{dx}(\sin x) = \cos x$

9) $\frac{d}{dx}(\cos x) = -\sin x$

10) $\frac{d}{dx}(\tan x) = \sec^2 x$

11) $\frac{d}{dx}(\cot x) = \operatorname{cosec}^2 x$

12) $\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

13) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cdot \cot x$

14) $\frac{d}{dx}(a^x) = a^x \cdot \log a$

15) $\frac{d}{dx}(\log a^x) = \frac{1}{a \cdot \log x}$

16) $\frac{d}{dx}(e^{ax}) = a \cdot e^{ax}$

17) $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$

18) $\frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{-2}{x^3}$

19) $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$

20) $\frac{d}{dx} \cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$

21) $\frac{d}{dx} \tan^{-1} x = \frac{1}{x^2+1}$

22) $\frac{d}{dx} \cot^{-1} x = \frac{-1}{x^2+1}$

23) $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

24) $\frac{d}{dx} \operatorname{cosec}^{-1} x = \frac{1}{x\sqrt{x^2-1}}$

25) $\frac{d}{dx}(u+v) = \frac{d}{dx}(u) + \frac{d}{dx}(v)$

26) $\frac{d}{dx}(u-v) = \frac{d}{dx}(u) - \frac{d}{dx}(v)$

27) $\frac{d}{dx}(uv) = u \cdot \frac{d}{dx}(v) + v \cdot \frac{d}{dx}(u)$

28) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot \frac{d}{dx}(u) - u \cdot \frac{d}{dx}(v)}{v^2}$

29) $\frac{d}{dx}(\sin ax) = a \cos x$

Integration formulas

1) $\int dx = x$

2) $\int x\,dx = \frac{x^2}{2}$

3) $\int x^n\,dx = \frac{x^{n+1}}{n+1}$

4) $\int \frac{1}{x} dx = \log x$

5) $\int \sin x \, dx = -\cos x$

6) $\int \cos x \, dx = +\sin x$

7) $\int \tan x \, dx = \log(\sec x)$

8) $\int a^x dx = \frac{a^x}{\log a}$

9) $\int \cot x \, dx = \log \tan x$

10) $\int \sec x \, dx = \log(\sec x + \tan x)$

11) $\int \operatorname{cosec} x \, dx = \log(\operatorname{cosec} x - \cot x)$

12) $\int \sec^2 x \, dx = \tan x$

13) $\int \operatorname{cosec}^2 x \, dx = -\cot x$

14) $\int e^x dx = e^x$

15) $\int e^{ax} dx = \frac{e^{ax}}{a}$

16) $\int \sin ax \, dx = -\frac{\cos ax}{a}$

17) $\int \sec x \cdot \tan x \, dx = \sec x$

18) $\int \operatorname{cosec} x \cdot \cot x \, dx = -\operatorname{cosec} x.$

Integration By parts formula.

$$\int uv \, dx = u\int v \, dx - \int\left\{\frac{d}{dx}(u)\cdot\int v \, dx\right\}dx + C$$

1) Find the derivative of the function $f(x) = 24x^3+12x^2-6$

Sol Given that

$f(x) = 24x^3+12x^2-6$

$f'(x) = \frac{d}{dx}[f(x)]$

$f'(x) = \frac{d}{dx}[24x^3+12x^2-6]$

$f'(x) = \frac{d}{dx}(24x^3)+\frac{d}{dx}(12x^2)-\frac{d}{dx}(6)$

$f'(x) = 24\cdot\frac{d}{dx}(x^3)+12\cdot\frac{d}{dx}(x^2)-6\cdot\frac{d}{dx}$

$f'(x) = 24(3.x^{3-1})+12(2x^{2-1})-6(0)$ $\left(\because \frac{d}{dx}x^n = n.x^{n-1}\right)$

$f'(x) = 24(3x^2)+12(2x)$

$f'(x) = 72x^2+24x$

2) $f(x) = 2x^3-6x^2-18x+21$

Sol Given that

$f(x) = 2x^3-6x^2-18x+21$

$f'(x) = \frac{d}{dx}[f(x)]$

$f'(x) = \frac{d}{dx}[2x^3-6x^2-18x+21]$

$f'(x) = \frac{d}{dx}(2x^3)-\frac{d}{dx}(6x^2)-\frac{d}{dx}(18x)+\frac{d}{dx}(21)$

$f'(x) = 2\cdot\frac{d}{dx}(x^3)-6\frac{d}{dx}(x^2)-18\frac{d}{dx}(x)+21\cdot\frac{d}{dx}$

$$f^1(x) = 2\cdot(3x^2) - 6(2x^1) - 18$$

$$f^1(x) = 6x^2 - 12x - 18$$

3) find $f^1(x) = x^6 - 6x^5 + \frac{5}{x^2} - \log x$

sol) $f(x) = x^6 - 6x^5 + \frac{5}{x^2} - \log x$

$$f^1(x) = \frac{d}{dx}(x^6) - 6\frac{d}{dx}(x^5) + 5\frac{d}{dx}\left(\frac{1}{x^2}\right) - \frac{d}{dx}\log x$$

$$f^1(x) = 6(x^{6-1}) - 6\cdot(5x^{5-1}) + 5\left(\frac{-2}{x^3}\right) - \frac{1}{x}$$

$$f^1(x) = 6x^5 - 30x^4 - \frac{10}{x^3} - \frac{1}{x}$$

4) $f(x) = \frac{8}{x^8} - \frac{6}{x^6} + 2$

sol) $f^1(x) = \frac{d}{dx}[f(x)]$

$$f^1(x) = \frac{d}{dx}\left(\frac{8}{x^8} - \frac{6}{x^6} + 2\right)$$

$$f^1(x) = 8\frac{d}{dx}\left(\frac{1}{x^8}\right) - 6\frac{d}{dx}\left(\frac{1}{x^6}\right) + 2\cdot\frac{d}{dx}(1)$$

$$f^1(x) = 8\left(\frac{-8}{x^9}\right) - 6\left(\frac{-6}{x^7}\right) + 2(0)$$

$$f^1(x) = \frac{-64}{x^9} + \frac{36}{x^7}$$

5) $f(x) = e^x + x^2 \cdot \sec x$

$f'(x) = \frac{d}{dx}[f(x)]$

$f'(x) = \frac{d}{dx}[e^x + x^2 \cdot \sec x]$

$f'(x) = \frac{d}{dx}(e^x) + \frac{d}{dx}(\overset{u}{x^2} \cdot \overset{v}{\sec x})$ $(\because \frac{d}{dx}(uv)$ formula).

$f'(x) = e^x + x^2 \cdot \frac{d}{dx}(\sec x) + \sec x \cdot \frac{d}{dx}(x^2)$

$f'(x) = e^x + x^2(\sec x \cdot \tan x) + \sec x\, 2 \cdot x^{2-1}$

$f'(x) = e^x + x^2 \sec x \cdot \tan x + 2 \sec x \cdot x$

6) $f(x) = x^3 \cdot \tan x$. $(\because \frac{d}{dx}(uv) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx})$

Sol $f'(x) = \frac{d}{dx}(x^3 \cdot \tan x)$

$\Rightarrow x^3 \frac{d}{dx}(\tan x) + \tan x \cdot \frac{d}{dx}(x^3)$ $(\because \frac{d}{dx} x^n = n \cdot x^{n-1})$

$\Rightarrow x^3 \sec^2 x + \tan x (3x^{3-1})$

$f'(x) \Rightarrow x^3 \sec^2 x + 3 \tan x \cdot x^2$

7) $f(x) = \sin 3x$

$f'(x) = \frac{d}{dx}(f(x))$ $\{\because \frac{d}{dx} \sin ax = a \cdot \cos x\}$

$f'(x) = \frac{d}{dx}(\sin 3x)$

$f'(x) = 3 \cos x$.

8) $f(x) = \left[\frac{3x+1}{-7x+5}\right]$ $\qquad \therefore \frac{d}{dx}\left(\frac{U}{V}\right) = \frac{V\frac{d}{dx}(u) - u\frac{d}{dx}(V)}{V^2}$

Here $u = 3x+1$, $7x+5 = V$

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,du}{dx} - u\frac{d}{dx}(v)$$

$$f'(x) = \frac{(-7x+5)\frac{d}{dx}(3x+1) - (3x+1)\frac{d}{dx}(7x+5)}{(-7x+5)^2}$$

$$= \frac{d}{dx}(3x+1) \qquad , \qquad \frac{d}{dx}(7x+5)$$

$$= \frac{d}{dx}(3x) + \frac{d}{dx}(1) \qquad \frac{d}{dx}(-7x) + \frac{d}{dx}(5)$$

$$= 3(1) \qquad = 7(1)$$

$$\frac{d}{dx}(3x+1) = 3 \qquad \frac{d}{dx}(7x+5) = 7$$

$$f'(x) = \frac{(-7x+5)(3) - (3x+1)(7)}{(-7x+5)^2}$$

$(\because (a+b)^2 = a^2+b^2+2ab)$

$$f'(x) = \frac{21x+15-21x+7}{49x^2+25+70x}$$

$$f'(x) = \frac{8}{49x^2+25+70x}$$

9) $f(x) = \left[\dfrac{x^4 - 2x + 3}{x^2}\right]$ $\left(\because \dfrac{d}{dx}\left(\dfrac{u}{v}\right) = v\dfrac{du}{dx} - u\dfrac{dv}{dx}\right)$

Sol) $f'(x) = \dfrac{x^2 \frac{d}{dx}(x^4 - 2x + 3) - (x^4 - 2x + 3)\frac{d}{dx}(x^2)}{(x^2)^2}$

$f'(x) = \dfrac{x^2 \frac{d}{dx}(x^4) - \frac{d}{dx}(2x) + \frac{d}{dx}(3) - (x^4 - 2x + 3)\frac{d}{dx}(x^2)}{(x^2)^2}$

$f'(x) = \dfrac{x^2[4x^3 - 2 + 0] - [x^4 - 2x + 3](2x)}{x^4}$

$f'(x) = \dfrac{(4x^5 - 2x^2) - (2x^5 - 4x^2 + 6x)}{x^4}$

$f'(x) = \dfrac{4x^5 - 2x^2 - 2x^5 + 4x^2 - 6x}{x^4}$

$f'(x) = \dfrac{2(x^5 + x^2 - 3x)}{x^4}$

$f'(x) = \dfrac{2x^5 + 2x^2 - 6x}{x^4}$

Maxima & Minima :

Steps for finding Maxima & Minima

step 1 :- find $f'(x)$ to the given function.

step 2 :- find the roots for $f'(x) = 0$

step 3 :- find $f''(x)$

step 4 :- Substitute x value in $f(x)$ and $f''(x)$

1. Find Maxima & Minima of $f(x) = 2x^3 - 6x^2 - 18x + 21$

Sol

$$f(x) = 2x^3 - 6x^2 - 18x + 21$$

Step 1 :- find $f'(x)$

$$f'(x) = \frac{d}{dx} f(x)$$

$$= \frac{d}{dx} (2x^3 - 6x^2 - 18x + 21)$$

$$f'(x) = 6x^2 - 12x - 18$$

Step 2 :- find roots $f'(x) = 0$

$$f'(x) = 6x^2 - 12x - 18$$

$$= 6x^2 - 12x - 18 = 0$$

$$= 6(x^2 - 2x - 3) = 0$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow x^2 + x - 3x - 3 = 0$$

$$\Rightarrow x(x+1) - 3(x+1) = 0$$

$$\Rightarrow (x-3)(x+1) = 0$$

$x-3=0 \quad , \quad x+1=0$

$x=3 \quad , \quad x=-1$

step 3 :- find $f''(x)$

$f'(x) = 6x^2-12x-18$

$f''(x) = \frac{d}{dx} f'(x)$

$f''(x) = 12x-12$

step 4 :- Substitute 'x' value in $f(x)$ & $f''(x)$

case (i) If $x=3$ sub in $f(x)$

$f(x) = 2x^3-6x^2-18x+21$

$= 2(3)^3-6(3)^2-18(3)+21$

$= 2(27)-6(9)-54+21$

$= \cancel{54}-\cancel{54}-54+21$

$f(3) = -33$

$f''(x) = 12x-12$

$f''(3) = 12(3)-12$

$f''(3) = 36-12$

$f''(3) = 24$

Case (ii) If $x=-1$ Sub in $f(x)$

$f(x) = 2(-1)^3-6(-1)^2-18(-1)+21$

$= 2(-1)-6(1)+18+21$

$= -2-6+39$

$= -8+39$

$f(x) = 31$

$f''(x) = 12x - 12$

$= 12(1) - 12$

$f''(-1) = -24$

$\therefore$ Maxima of $f(x) = 31$

$\therefore$ Minima of $f(x) = -33$

2) $f(x) = 4x^3 - 18x^2 + 24x - 7$

Sol Step 1:- To find $f'(x)$

$f'(x) = \frac{d}{dx}[f(x)]$

$= \frac{d}{dx}(4x^3 - 18x^2 + 24x - 7]$

$f'(x) = 12x^2 - 36x + 24$

Step 2 :- find root of $f'(x) = 0$

$f'(x) = 12x^2 - 36x + 24 = 0$

$= 12(x^2 - 3x + 2) = 0$

$= x^2 - 3x + 2 = 0$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore a = 1, b = -3, c = 2$

$x = \frac{3 \pm \sqrt{9 - 4(1)(2)}}{2(1)}$

$x = \frac{3 \pm \sqrt{9-8}}{2}$

$x = \frac{3 \pm \sqrt{1}}{2}$

$x = \frac{3 \pm 1}{2}$

$\therefore\ x = \frac{3+1}{2}$, $x = \frac{3-1}{2}$

$x = \frac{4}{2}$, $x = \frac{2}{2}$

$x = 2$, $x = 1$

step 3 :- find $f''(x)$

$$f''(x) = \frac{d}{dx}\left[f'(x)\right]$$

$$= \frac{d}{dx}\left[12x^2 - 36x + 24\right]$$

$$f''(x) = 24x - 36$$

step 4 :-

case(i) If $x=1$, sub in $f(x)$ & $f''(x)$

$$f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$= 4(1) - 18(1)^2 + 24(1) - 7$$

$$= 4 - 18 + 24 - 7$$

$$= 28 - 25$$

$$f(1) = 3$$

$$f''(x) = 24(x) - 36$$

$$f''(1) = 24(1) - 36$$

$$= 24 - 36$$

$$f''(1) = -12$$

case(ii) If $x=2$, sub in $f(x)$ & $f''(x)$

$$f(x) = 4x^3 - 18x^2 + 24x - 7$$

$$f(2) = 4(2)^3 - 18(2)^2 + 24(2) - 7$$

$$= 4(8) - 18(4) + 48 - 7$$

$$= 32 - 72 + 48 - 7$$

$$f(2) = 80 - 79$$

$f(2) = 1$

$f''(x) = 24(x) - 36$

$= 24(2) - 36$

$= 48 - 36$

$f''(x) = 12$

$\therefore$ Maxima of $f(x) = 12$

$\therefore$ Minima of $f(x) = -12$.

3) $f(x) = 2x^3 + 3x^2 - 12x + 3$

Sol Step 1:- To find $f'(x)$

$f'(x) = \frac{d}{dx} f(x)$

$= \frac{d}{dx}(2x^3 + 3x^2 - 12x + 3)$

$f'(x) = 6x^2 + 6x - 12$

Step 2:- To find root of $f'(x) = 0$

$f'(x) = 0$

$f'(x) = 6x^2 + 6x - 12x = 0$

$= 6(x^2 + x - 2)$

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$a = 1,\ b = 1,\ c = -2$

$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)}$

$$x = \frac{-1 \pm \sqrt{1+8}}{2}$$

$$= \frac{-1 \pm \sqrt{9}}{2}$$

$$= \frac{-1 \pm 3}{2}$$

$x = \frac{-1+3}{2}$, $x = \frac{-1-3}{2}$

$x = \frac{2}{2}$, $x = -\frac{4}{2}$

$x = 1$, $x = -2$

step 3:- find $f''(x)$

$$f''(x) = \frac{d}{dx}(f'(x))$$

$$= \frac{d}{dx}(6x^2+6x-12)$$

$$f''(x) = 12x+6$$

step 4:-

case (i) If $x=1$ sub in $f(x)$ & $f''(x)$

$$f(x) = 2x^3+3x^2-12x+3$$

$$f(1) = 2(1)^3+3(1)^2-12(1)+3$$

$$= 2+3-12+3$$

$$f(1) = -4$$

$$f''(x) = 12x+6$$

$$= 12(1)+6$$

$$f''(x) = 18$$

case(ii) if $x = -2$ sub in $f(x)$ & $f''(x)$

$$f(x) = 2x^3 + 3x^2 - 12x + 3$$
$$= 2(-2)^3 + 3(-2)^2 - 12(-2) + 3$$
$$= 2(-8) + 3(4) + 24 + 3$$
$$= -16 + 12 + 27$$
$$f(-2) = 23$$

$$f''(x) = 12x + 6$$
$$= 12(-2) + 6$$
$$= -24 + 6$$
$$f''(-2) = -18$$

$\therefore$ Maxima of $f(x) = 23$

Minima of $f(x) = -18$

for practice :-

$$f(x) = 2x^3 - 9x^2 + 12x + 5$$

Integration problems :-

1. Evaluate $\int \sqrt{x}\, dx$

Sol) $\int \sqrt{x}\, dx$ $\qquad (\because \int x^n dx = \frac{x^{n+1}}{n+1})$

$$= \int x^{1/2}\, dx$$

$$= \frac{x^{1/2+1}}{1/2+1}$$

$$= \frac{x^{3/2}}{\frac{3}{2}}$$

$$= \frac{2x^{3/2}}{3}$$

2) find $\int \frac{1}{\sqrt{x}}\,dx$

$$= \int x^{-1/2}\,dx$$

$$= \frac{x^{-1/2+1}}{-\frac{1}{2}+1}$$

$$= \frac{x^{1/2}}{1/2}$$

$$= 2x^{1/2}$$

$$\int \frac{1}{\sqrt{x}}dx \quad 2\sqrt{x}$$

3) $\int e^{-3x}\,dx$

$$\int e^{-3x}\,dx = \frac{e^{-3x}}{3} \qquad \left(\because \int e^{ax}dx = \frac{e^{ax}}{a}\right)$$

4) $\int 3^x\,dx$

$$\int 3^x\,dx = \frac{3^x}{\log 3} \qquad \left(\because \int a^x\,dx = \frac{a^x}{\log a}\right)$$

5) $\int (x^3+3^x+2)\,dx$

$= \int x^3\,dx + \int 3^x\,dx + \int 2\,dx$ $\quad (\because \int x^n dx$

$\because \int a^x dx$

$\because \int dx$

$= \dfrac{x^4}{4} + \dfrac{3^x}{\log 3} + 2x$

6) $\int (3x^2+2x+5)\,dx$

$\int (3x^2+2x+5)\,dx = \int 3x^2 dx + \int 2x\,dx + \int 5\,dx$

$\Rightarrow 3\int x^2 dx + 2\int x\,dx + 5\int dx$

$\Rightarrow \cancel{3}\cdot\dfrac{x^3}{\cancel{3}} + \cancel{2}\cdot\dfrac{x^2}{\cancel{2}} + 5x$

$\Rightarrow x^3+x^2+5x$

7) $\int (\sec^2 x - e^x + \sin x)\,dx$

$\int (\sec^2 x - e^x + \sin x)\,dx = \int \sec^2 x\,dx - \int e^x dx + \int \sin x\,dx$

$= \tan x - e^x - \cos x.$

8) $\int (1-x)(4-3x)(3+2x)\,dx$

Sol $\int (4-3x-4x+3x^2)(3+2x)\,dx$

$\int (4-7x+3x^2)(3+2x)\,dx$

$\int (12+8x-21x-14x^2+9x^2+6x^3)\,dx$

$\int (12-13x-5x^2+6x^3)\,dx$

$= \int (6x^3-5x^2-13x+12)\,dx$

$= 6\int x^4\,dx - 5\int x^3\,dx - 13\int x\,dx + 12\int dx$

$= 6\frac{x^4}{4} - 5\frac{x^3}{3} - 13\frac{x^2}{2} + 12x$

9) $\int (x+1)(x-2)(2x+3)\,dx$

$= \int (x^2 - 2x + x - 2)(2x+3)\,dx$

$= \int (x^2 - x - 2)(2x+3)\,dx$

$= \int (2x^3 + 3x^2 - 2x^2 - 3x - 4x - 6)\,dx$

$= \int (2x^3 + x^2 - 7x - 6)\,dx$

$= \int 2x^3\,dx + \int x^2\,dx - 7\int x\,dx - 6\int dx$

$= \frac{2x^4}{4} + \frac{x^3}{3} - 7\frac{x^2}{2} - 6x$

10) Homework. $\int (1-x)^2 (2+x)\,dx$.

11) $\int \left(x + \frac{1}{x}\right)^2 dx$

12) $\int \left(x + \frac{1}{x}\right)^3 dx$ $\rightarrow (\because (a+b)^3$ formula.

$= \int \left(x^3 + 3\cdot \cancel{x^2}\cdot\frac{1}{\cancel{x}} + 3\cdot\cancel{x}\cdot\frac{1}{\cancel{x^2}} + \frac{1}{x^3}\right) dx$

$= \int \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) dx$

$= \int (x^3\,dx + 3\int x + 3\int \frac{1}{x} + \int \frac{1}{x^3}\,dx$

$$= \frac{x^4}{4} + 3\frac{x^2}{2} + 3\log x + \int x^{-3}\,dx$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\log x + \frac{x^{-3+1}}{-3+1}$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\log x + \frac{x^{-2}}{-2}$$

$$= \frac{x^4}{4} + \frac{3x^2}{2} + 3\log x - \frac{1}{2x^2}$$

13) Find $\int \left(\frac{x^2+2x+3}{x^4}\right) dx$

Sol

$$= \int \left(\frac{x^2+2x+3}{x^4}\right) dx$$

$$= \int \left(\frac{\cancel{x^2}}{x^{\cancel{4}}} + \frac{2\cancel{x}}{x^{\cancel{4}}} + \frac{3}{x^4}\right) dx$$

$$= \int \left(\frac{1}{x^2} + \frac{2}{x^3} + \frac{3}{x^4}\right) dx$$

$$= \int \frac{1}{x^2}\,dx + 2\int \frac{1}{x^3}\,dx + 3\int \frac{1}{x^4}\,dx$$

$$= \int x^{-2}\,dx + 2\int x^{-3}\,dx + 3\int x^{-4}\,dx$$

$$= \frac{x^{-2+1}}{-2+1} + 2\cdot\frac{x^{-3+1}}{-3+1} + 3\cdot\frac{x^{-4+1}}{-4+1}$$

$$= \frac{x^{-1}}{-1} + \cancel{2}\cdot\frac{x^{-2}}{-\cancel{2}} + \cancel{3}\frac{x^{-3}}{-\cancel{3}}$$

$$= -\frac{1}{x} - \frac{1}{x^2} - \frac{1}{x^3}$$

Integration by parts :-

formula:- $\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx + c$

1) Evaluate $\int x\cos x\,dx$.

let $u = x$, $v = \cos x$

$\therefore \int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx + c$

$$\int x\cos x\,dx = x\int \cos x\,dx - \int\left\{\frac{d}{dx}(x)\cdot\int \cos x\,dx\right\}dx + c$$

$$= x(\sin x) - \int 1\cdot\sin x\,dx + c$$

$$= x\sin x - \int \sin x\,dx + c$$

$$= x\sin x - (-\cos x) + c$$

$$= x\sin x + \cos x + c$$

$\therefore \int x\cos x\,dx = x\sin x + \cos x + c$.

2) practice $\int x\sin x\,dx$.

3) $\int x\cdot e^x\,dx$

Sol let $u = x$, $v = e^x$

$$\int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\int v\,dx\right\}dx + c$$

$$= x\int e^x\,dx - \int\left\{\frac{d}{dx}(x)\cdot\int e^x\,dx\right\}dx + c$$

$$= x\cdot e^x - \int 1\cdot e^x\,dx + c$$

$$= x\cdot e^x - e^x + c$$

$\int x\cdot e^x\,dx = e^x(x-1) + c$

4) $\int x^2 \cdot e^{2x} dx$ for pratice.

5) $\int x \log x \, dx$

Here $u = \log x$, $v = x$.

$\therefore \int uv\,dx = u\int v\,dx - \int\left\{\frac{d}{dx}(u)\cdot\int v\,dx\right\}dx + c$

$\therefore \int \log x \cdot x\,dx = \log x \int x\,dx - \int\left\{\frac{d}{dx}(\log x)\cdot\int x\,dx\right\}dx + c$

$= \log x \cdot \frac{x^2}{2} - \int\left[\frac{1}{x}\cdot\frac{x^2}{2}\right]dx + c$

$= \log x \cdot \frac{x^2}{2} - \int\left(\frac{x}{2}\right)dx + c$

$= \log x \cdot \frac{x^2}{2} - \frac{1}{2}\int x\,dx + c$

$= \log x \cdot \frac{x^2}{2} - \frac{1}{2}\cdot\frac{x^2}{2} + c$

$\therefore \int \log x \cdot x\,dx = \log x \cdot \frac{x^2}{2} - \frac{x^2}{4} + c$

Managerial Applications of Differentiation and Integration

Differentiation and integration are essential tools in managerial applications, providing insights into the behavior of business functions and helping to make informed decisions. Here are some key applications:

diffrentiaton applications

1. Profit Maximization

Differentiation helps businesses determine the level of output at which profit is highest by comparing how revenue and cost change with each additional unit produced.

2. Cost Control and Efficiency

It allows managers to understand how costs increase as production increases, helping them identify the most efficient level of production.

3. Pricing Decisions

Differentiation helps firms study how quantity demanded changes when price changes. This supports making smart pricing decisions.

4. Revenue Management

By analyzing the rate at which revenue changes with sales, managers can determine the most beneficial sales level and identify when selling more becomes less profitable.

5. Optimization in Operations

It is used to find minimum cost or maximum output levels in areas like production, transportation, and resource allocation.

6. Sensitivity and Risk Analysis

Differentiation shows how sensitive profit, cost, or revenue is to changes in variables like price, demand, or expenses.

Integration applications

1. Total Cost Estimation

Integration helps businesses calculate total production cost when they know how cost increases gradually with each unit.

2. Total Revenue Calculation

When the change in revenue is known, integration helps determine the overall revenue earned over a range of sales.

3. Inventory and Supply Chain Planning

Integration helps determine total inventory levels, total demand, and total production over a period, which is critical for logistics and supply chain management.

4. Market Analysis (Consumer and Producer Surplus)

Integration is used to measure the overall benefit consumers receive from purchasing products and the benefit producers gain by selling above their minimum acceptable price.

5. Financial Valuation

It helps calculate the total value of income or cash flow received continuously over time, which is useful in investment and budgeting decisions.

6. Productivity and Learning Curves

Integration helps determine total labor or production when workers become more efficient over time.

Unit-3

PROGRESSIONS

Progression :

A progression (which is also known as a sequence) is nothing but a pattern of numbers. For example, 3, 6, 9, 12, ... is a progression because there is a pattern observed where every number here is obtained by adding 3 to its previous number. But this pattern doesn't need to be the same in every progression.

The pattern of a progression depends on its type. Let us see the types of progressions along with examples and their formulas.

What is the Definition of a Progression?

Progression is a list of numbers (or items) that exhibit a particular pattern. A progression is also known as a sequence. In a progression, every term is obtained by applying a specific rule on its previous term. In other words, every term of a progression is defined by a general term (or) n^{th} term, which is denoted by a_n.

For example, the n^{th} term of a progression 3,5,7,9, ... is $a_n = 2n + 1$. Substituting n = 1,2,3, ... here, we get the 1^{st}, 2^{nd}, 3^{rd}, ... terms. For example:

- When n = 1, first term = 2(1) + 1 = 3.
- When n = 2, second term = 2(2) + 1 = 5.
- When n = 3, third term = 2(3) + 1 = 7

  and so on.

Types of Progressions

There are mainly 3 types of progressions in math. They are:

- Arithmetic Progression (AP)
- Geometric Progression (GP)
- Harmonic Progression (HP)

Each type of progression along with a simple definition and example is tabulated below.

Progression	Definition	Example
Arithmetic Progression (AP)	The differences between any two consecutive numbers are all same.	1, 4, 7, 10, ...
Geometric Progression (GP)	The ratios of any two consecutive numbers are all same.	4, 16, 64, 256, ...
Harmonic Progression (HP)	The reciprocals of terms form an AP.	1/2, 1/4, 1/6, ...

Arithmetic Progression

An arithmetic progression (AP) is a sequence of numbers in which each successive term is the sum of its preceding term and a fixed number. This fixed number is called the common difference. For example, 1, 4, 7, 10, ... is an AP as every number is obtained by adding a fixed number 3 to its previous term.

- **2^{nd} term** = 4 = 1 + 3 = **1^{st} term + 3**
- **3^{rd} term** = 7 = 4 + 3 = **2^{nd} term + 3**
- **4^{th} term** = 10 = 7 + 3 = **3^{rd} term + 3**

  and so on.

Arithmetic Progression Formulas

Let the first term of the progression be a, the common difference be d, and the n^{th} term be a_n. Then, the arithmetic progression formulas are given by:

- $a_n = a + (n - 1) d$
- $d = a_n - a_{n-1}$
- Sum of the first n terms, $S_n = n/2(2a+(n-1)d)$ (or) $S_n = n/2(a + l)$, where l = the last term = T_n.

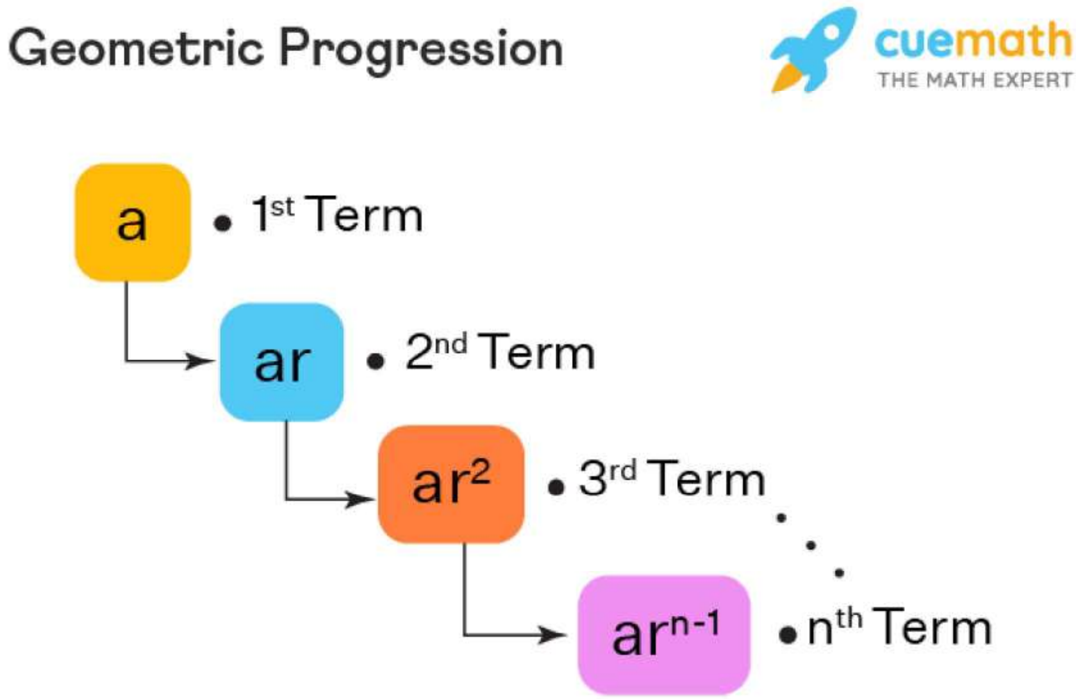
Geometric Progression :

A geometric progression (GP) is a sequence of numbers in which each successive term is the product of its preceding term and a fixed number. This fixed number is called the common ratio. For example, 4, 16, 64, 256, ... is a GP as every number is obtained by multiplying a fixed number 4 to its previous term.

- **2^{nd} term** = 16 = 4(4) = **4(1^{st} term)**
- **3^{rd} term** = 64 = 4(16) = **4(2^{nd} term)**
- **4^{th} term** = 256 = 4(64) = **4(3^{rd} term)**

  and so on.

In general, a geometric progression looks like this:



Here,

- 'a' is the first term and
- 'r' is the common ratio (fixed number)

Geometric Progression Example

Consider an example of a geometric progression: 1, 4, 16, 64, ... Observe that 4/1 = 16/4 = 64/16 = ... = 4. All the ratios are same. Hence it is a GP.

Geometric Progression Formulas

Let the first term of the progression be a, the common ratio be r, and the n^{th} term be a_n. Then, the geometric progression formulas are given by:

- $a_n = ar^{n-1}$
- Sum of the first 'n' terms, $S_n = a(r^n - 1) / (r - 1)$ when $r \neq 1$ and $S_n = na$ when $r = 1$.
- Sum of infinite geometric series, $S_\infty = a / (1 - r)$ when $|r| < 1$ and S_∞ diverges when $|r| \geq 1$.

Harmonic Progression

A harmonic progression is a sequence obtained by taking the reciprocal of the terms of an arithmetic progression. The sequence of natural numbers is an arithmetic progression. So, taking reciprocals of each term we get 1,1/2,1/3,1/4,... This is harmonic progression.

Harmonic Progression Example

When a ball is dropped, the initial height reached by the ball is 1/2 units and after the first impact, the height attained by the ball is 1/4 units. After the second impact, the height attained by the ball is 1/6 units. After the third impact, the height attained by the ball is 1/8 units, and so on. Now the sequence of heights formed by the ball is: 1/2, 1/4, 1/6, 1/8, ...

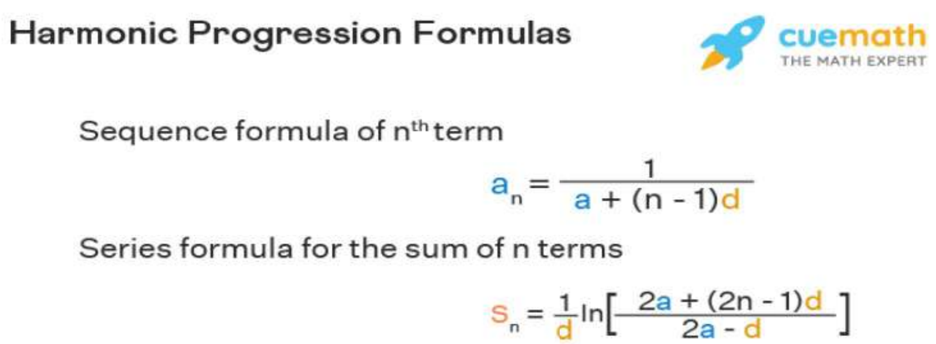
This sequence is a harmonic progression because the reciprocal of all the terms of this progression form an arithmetic progression.

- Reciprocal of the sequence: 2, 4, 6, 8, ... → AP
- Hence, the sequence: 1/2, 1/4, 1/6, 1/8, ... → HP

Harmonic Progression Formulas

For a harmonic progression 1/a, 1/(a+d), 1/(a+2d), ...

- n^{th} term, $a_n = 1 / (a + (n - 1) d)$
- Sum of the first n terms, $S_n = 1/d \ln [(2a + (2n - 1) d] / (2a - d)]$



Applications of Arithmetic Progression (AP)

- **Finance:** Calculating the total amount saved after adding a fixed amount of money each month.
- **Physics:** Modeling linear phenomena like constant speed, such as the time intervals between successive bus arrivals.
- **Design and Aesthetics:** Creating patterns in designs like rangoli, embroidery, or flower arrangements, where rows increase or decrease by a constant number of items.
- **Engineering:** Calculating straight-line depreciation.

Applications of Geometric Progression (GP)

- **Finance:** Calculating compound interest or the value of investments that grow exponentially.
- **Biology:** Modeling the exponential growth of populations like bacteria or the decay of radioactive substances.
- **Physics:** Studying the decay of radioactive materials.
- **Economics:** Analyzing trends of technological advancement that have an exponential growth rate.

Applications of Harmonic Progression (HP)

- **Physics:** Used in wave motion and oscillation studies, such as calculating the frequency of a harmonic series.
- **Engineering:** Calculating the effective resistance of resistors in parallel, where the formula is

  1/R=1/R1+1/R2+...1 / cap R equals 1 / cap R sub 1 plus 1 / cap R sub 2 plus point point point

  $1/R=1/R1+1/R2+....$

- **Optics:** Calculating the relationship between object distance, image distance, and focal length of lenses and mirrors.
- **Work Rate:** Finding the combined time it takes for two people to complete a job together, based on their individual times.

Annuity and present values

Theory of annuities and present values

1. Annuities — Basic Concept

An **annuity** is a sequence of equal payments made at regular, equally spaced time intervals. Common examples include loan repayments, pension payments, rent, or savings deposits.

Types of annuities

1. **Ordinary Annuity**
   - Payments are made **at the end** of each period.
2. **Annuity Due**
   - Payments are made **at the beginning** of each period.
3. **General Annuity**
   - Payment intervals and interest compounding periods may differ.

2. Present Value of an Annuity

The **present value (PV)** of an annuity is the value today of a series of future payments, discounted using an interest rate.

The idea is that money received in the future is worth **less** than money received today because of the time value of money.

PV of Ordinary Annuity (Theory)

The present value is found by discounting each periodic payment back to the present using compound interest.
Since the payments are equal and equally spaced, the formula sums a geometric series of discounted amounts.

3. Annuities in Progressions

An annuity does not have to consist of equal payments. The payments can follow a **progression**, usually arithmetic or geometric.

A. Annuity in Arithmetic Progression (AP)

- Payments increase or decrease by a **constant amount**, called the common difference.
- The sequence:

  R, R+d, R+2d, …, R+(n−1)dR,\; R + d,\; R + 2d,\; \dots,\; R + (n-1)dR,R+d,R+2d,…,R+(n−1)d

- To find the present value, each term of the arithmetic series is discounted to the present.
- The PV formula combines the PV of a level annuity and an adjustment term for the arithmetic increase/decrease.

B. Annuity in Geometric Progression (GP)

- Payments increase or decrease by a **constant percentage**, called the common ratio.
- The sequence:

R, Rg, Rg2, …, Rgn−1R,\; Rg,\; Rg^2,\; \dots,\; Rg^{n-1}R,Rg,Rg2,…,Rgn−1

- The present value is obtained by discounting each term of the geometric progression.
- Because discounting also creates a geometric sequence, the PV formula uses the ratio g1+i\frac{g}{1+i}1+ig.

Nature: Exponential growth or decline in payments.

Fundamental Concepts Behind Present Values

- **Time Value of Money:** A future sum is worth less today because money can earn interest.
- **Discounting:** The process of converting future cash flows to their equivalent today using an interest rate.
- **Compounding:** The reverse process—growing present money to a future value.
- **Series Summation:** PV of an annuity relies on summing multiple discounted values, often forming a geometric series.

Importance of Annuity PV Theory

Present values are used to:

- Price financial products (loans, bonds, insurance policies)
- Evaluate investment opportunities
- Calculate pensions and retirement plans
- Compare streams of cash flows occurring over time

Examples

Calculating the Present Value of an Annuity

The formula for the present value of an ordinary annuity is below. An ordinary annuity pays interest at the end of a particular period, rather than at the beginning:

P=PMT×*r*1−((1+*r*)*n*1)

where:P=Present value of an annuity stream PMT=Dollar amount of each annuity payment r=Interest rate (also known as discount rate) n=Number of periods in which payments will be made

Real-World Example: Assessing Annuity Present Value

Assume a person has the opportunity to receive an ordinary annuity that pays $50,000 per year for the next 25 years, with a 6% discount rate, or take a $650,000 lump-sum payment. Which is the better option? Using the above formula, the present value of the annuity is:

Present value=$50,000×1−(1(1+0.06)25)0.06=$639,168Present value =$50,000×0.061−((1+0.06)251) =$639,168

Given this information, the annuity is worth $10,832 less on a time-adjusted basis, so the person would come out ahead by choosing the lump-sum payment over the annuity.

1. For AP is $\frac{1}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{-5}{4}$ ...... write for AP the first term a and common difference d also find 7^{th} term.

Here,

$$\frac{1}{4}, \frac{-1}{4}, \frac{-3}{4}, \frac{-5}{4}$$

$$a = \frac{1}{4} \quad d = t_2 - t_1$$

$$= \frac{-1}{4} - \frac{1}{4}$$

$$= \frac{-\cancel{2}^1}{\cancel{4}_2} = -1/2$$

$$d = -1/2$$

n^{th} term in AP

$$a_n = a + (n-1)d$$

$$a_7 = \frac{1}{4} + (7-1)(-1/2)$$

$$= \frac{1}{4} + (6)(-1/2)$$

$$a_7 = \frac{1}{4} + (\cancel{6}_3)(-1/\cancel{2}_1)$$

$$= \frac{1}{4} - 3$$

$$= \frac{1-12}{4}$$

$$= -11/4$$

2. Find the 10^{th} term of the AP 5, +1, -3, -7

AP = 5, 1, -3, -7

$a = 5$; $d = t_2 - t_1$

$= 1 - 5$

$d = -4$

n^{th} term of AP

$a_n = a + (n-1)d$

$a_{10} = 5 + (10-1) - 4$

$= 5 + (9) - 4$

$= 5 + (-36)$

$= 5 - 36$

$a_{10} = -31$

3. which term of the AP 3, 8, 13, 18, ... is 78

AP = 3, 8, 13, 18 .....

Here $a = 3$; $d = t_2 - t_1$

$= 8 - 3$

$= 5$

$a_n = a + (n-1)d$

$78 = 3 + (n-1)5$

$78 = 3 + (5n - 5)$

$78 = -2 + 5n$

$\cancel{80}^{16} = \cancel{5}^{1} n$

$n = 16$ //

4. Determine the AP whose 3rd term is 5 and 7th term is 9.

$a_3 = a + (3-1)d = 5$

$= a + 2d = 5$ —— ①

$a_7 = a + (7-1)d = 9$

$= a + 6d = 9$ —— ②

By solving ① and ② equations

$$\cancel{a} + 2d = 5$$
$$\cancel{a} + 6d = 9$$
$$(-) \quad (-) \quad (-)$$
$$-\cancel{4}d = -\cancel{4}$$
$$d = 1$$

when $d = 1$ substitute in equation ①.

$a + 2d = 5$

$a + 2(1) = 5$

$a + 2 = 5$

$a = 3$ //

Required AP is $a, a+d, a+2d$

$3, 4, 5, 6, 7, 8, 9, 10 \ldots$

5. For what value of n are the n^{th} terms of two AP $63, 65, 67 \ldots$ and $3, 10, 17 \ldots$ equal

$63, 65, 67$ are in AP

Here $a = 63$ $d = t_2 - t_1$

$= 65 - 63$

$= 2$

$a_n = a + (n-1)d$

$a_n = 63 + (n-1)2$

$a_n = 63 + 2n - 2$

$a_n = 61 + 2n$

3, 10, 17 ..... are in AP

Here $a = 3$ $d = 10 - 3$

$d = 7$

$a_n = a + (n-1)d$

$= 3 + (n-1)7$

$= 3 + 7n - 7$

$= 7n - 4$

Given that the n^{th} term of two AP's equal.

$61 + 2n = 7n - 4$

$61 + 4 = 2n - 7n$

$\cancel{65}_{13} = \cancel{5}_{1} n$

$n = 13$

6. Find the 11^{th} term is AP is $-3, -1/2, 2$ ....

$a = -3$ $d = t_2 - t_1$

$d = -1/2 - 3$

$= -7/2$

$a_n = a + (n-1)d$

$a_{11} = -3 + (11-1) - 7/2$

$= -3 + (10) - 7/2$

$= -3 + (-35)$

$= -38$

7. Find 50^{th} term of AP is 1, 3, 5 ....

Here Given $a = 1$ $\quad d = t_2 - t_1$

$= 3 - 1$

$= 2$

50^{th} term = ?

$a^n = a + (n-1)d$

$= a + (50-1)2$

$= 1 + (49)2$

$= 1 + 98$

$= 99$

50^{th} term is 99

8. Find the first four terms in AP. where $a = 10$, $d = 10$ and also Find 14^{th} term.

Here Given $a = 10$; $d = 10$

First term $a = 10$

second term $a + d = 10 + 10 = 20$

third term $a + 2d = 10 + 2(10) = 30$

Fourth term $a + 3d = 10 + 3(10) = 40$

Now first four terms are $= 10, 20, 30, 40, 50 \ldots$

Now 14^{th} term is ___ ?

$$a_n = a + (n-1)d$$

$$= 10 + (14-1)10$$

$$= 10 + (13)10$$

$$= 10 + 130$$

$$a_{14} = 140$$

9. The sum of first 14 terms of an AP is 1050 and its first term is 10 then find 20^{th} term.

Given that $a = 10$

$S_n = 1050$

$n = 14^{th}$ term

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$S_n = \frac{14}{2}\left[2(10) + (14-1)d\right]$$

$$= 1050 = 7\left(2(10) + (14-1)d\right)$$

$$= 1050 = 7\left[20 + 13d\right]$$

$$= 1050 = 140 + 13d$$

$$910 = 91d$$

$$d = 10$$

20^{th} term $a_{20} = a + 19d$

$$= 10 + 19 \times 10$$

$$= 10 + 190$$

$$= 200$$

10 The sum 4^{th}, 8^{th} term is 24 and sum $0 \div 6^{th}$, 10^{th} is 44 then find first three terms of AP.

Given that

4^{th} term $a^4 = a + 3d$

8^{th} term $a^8 = a + 7d$

Given that of 4^{th} term, 8^{th} term is 24

$a + 3d + a + 7d = 24$

$2a + 10d = 24$

$a + 5d = 12$ —— ①

Given 6^{th} term $= a^6 = a + 5d$

10^{th} term $= a^{10} = a + 9d$

Given sum of a^6 and a^{10} is 44

$a + 5d + a + 9d = 44$

$2a + 14d = 44$

$a + 7d = 22$ —— ②

Now we solve ① and ② equation

$a + 5d = 12$

$a + 7d = 22$

$(-) \quad (-) \quad (-)$

$-\cancel{2}d = -\cancel{10}\,5$

$d = 5$

Now $d = 5$ will subtitute in equal ⑬

$$a + 5d = 12 \quad - ⑬$$

$$a + 5(5) = 12$$

$$a + 25 = 12$$

$$a = -13$$

Here $a = -13$, $d = 5$

First terms $= a = -13$

Second term $= a + d = -13 + 5 = -8$

third term $= a + 2d = -13 + 2(5) = -3$

the first three terms all $= -13, -8, -3$

Geomentric Progression - (Reciprocal to AP)

$$a, ar, ar^2 \dots ar^{n-1}$$

$$\frac{a^2}{a_1} = \frac{a^3}{a^2} = \frac{a^4}{a^3}$$

General form of GP is $a, ar, ar^2, \dots ar^{n-1}$

n^{th} term in GP $= \boxed{a_n = ar^{n-1}}$

$$r = \frac{a^2}{a_1}$$

1. Find the 20^{th} and n^{th} term of the GP $\frac{5}{2}, \frac{5}{4}, \frac{5}{8} \dots$

Given that $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}$ are in G.P

Here $a = \frac{5}{4}$; $r = \frac{a_2}{a_1} = \frac{5/4}{5/2}$

$$r = \frac{\cancel{5}}{\cancel{4}_2} \times \frac{\cancel{2}^1}{\cancel{5}}$$

$r = 1/2$

20^{th} term $= a^{20} = ar^{n-1}$

$= \frac{5}{2} (1/2)^{20-1}$

$= \frac{5}{2} (1/2)^{19}$

$= \frac{5}{2} \left(\frac{1^{19}}{2^{19}}\right)$

$= \frac{5}{2^{20}}$

n^{th} term $a^n = ar^{n-1}$

$= \frac{5}{2} (1/2)^{n-1} = \frac{5}{2} \left(\frac{1}{2^{n-1}}\right)$

$= \frac{5}{2^{n-\cancel{1}+\cancel{1}}} = \frac{5}{2^n}$ //

2. which term of the Gp is $2, 2\sqrt{2}, 4 \dots$ is 128

Given $2, 2\sqrt{2}, 4$

$a = 2$; $r = \frac{a_2}{a_1} = \frac{\cancel{2}\sqrt{2}}{\cancel{2}} = \sqrt{2}$

$a_n = 128 \quad n = ?$

$a^n = ar^{n-1}$

$\cancel{128}\ 64 = \cancel{2} \cdot (\sqrt{2})^{n-1}$

$64 = [(2)^{1/2}]^{n-1}$

$64 = 2^{\left(\frac{n-1}{2}\right)}$

$$2^6 = 2^{\left(\frac{n-1}{2}\right)}$$

$$6 = \frac{n-1}{2}$$

$$12 = n-1$$

$$n = 13 \,//$$

13^{th} term is 128

3. In a Gp the third term is 24 and 6^{th} term is 192 find the 10^{th} term ?

3^{rd} term in Gp = 24 ; 6^{th} term in Gp = 192

$a^3 = ar^{3-1} = 24$ $\qquad$ $a^6 = ar^{6-1} = 192$

$ar^2 = 24$ —— ① $\qquad$ $ar^5 = 192$ —— ②

Divided ② by ① $= \frac{\cancel{a}r^{\cancel{5}\,3}}{\cancel{a}r^{\cancel{2}}} = \frac{192}{24}$

$$r^3 = 8$$

$$r^3 = 2^3 = r = 2$$

$r = 2$ subtitute in ① $= ar^2 = 24$

$$= a(2)^2 = 24$$

$$= a(\cancel{4}^{1}) = \cancel{24}\,6$$

$$= a = 6$$

Now find the 10^{th} term

$$a10 = ar^9$$

$$= 6(2)^9$$

$$= 6(512) = 3072 .$$

4. Find the x value so that $x, x+2, x+6$ are consequent terms of GP.

Let $a_1 = x$; $a_2 = x+2$; $a_3 = x+6$

we now that $\frac{a_2}{a_1} = \frac{a_3}{a_2}$

$$\frac{x+2}{x} = \frac{x+6}{x+2}$$

$$(x+2)^2 = x(x+6)$$

$$x^2 + 2^2 + 2.x.2 = x^2 + 6x$$

$$\cancel{x^2} + 4 + 4x = \cancel{x^2} + 6x$$

$$2\cancel{4} = \cancel{2}x$$

$$x = 2$$

5. Find 4, 8, 16 .... write the again three terms of the Given GP

Given $a = 4$ $\quad r = \frac{a_2}{a_1} = \frac{\cancel{8}^2}{\cancel{4}_1} = 2$

$$r = 2$$

The required GP is

$$a, ar, ar^2, ar^3, ar^4, ar^5, \ldots$$

$$= 4, 4\times2, 4\times2^2, 4\times2^3, 4\times2^4, 4\times2^5$$

$$= 4, 8, 4\times4, 4\times8, 4\times16, 4\times32$$

$$= 4, 8, 16, 64, 128, 256$$

6. If $a=1$, $r=\sqrt{2}$ then find S_{20} in GP

Given that $a=1$

$$r=\sqrt{2}$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

$$S_{20} = \frac{1[(\sqrt{2})^{20}-1]}{\sqrt{2}-1}$$

$$= \frac{((2^{1/2})^{20}-1)}{\sqrt{2}-1}$$

$$= \frac{2^{10}-1}{\sqrt{2}-1}$$

$$= \frac{1024-1}{\sqrt{2}-1}$$

$$= \frac{1023}{\sqrt{2}-1} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{1023(\sqrt{2}+1)}{(\sqrt{2})^2-1^2}$$

$$= 1023(\sqrt{2}+1)$$

Harmonic progression

In $a_1, a_2, a_3, \ldots a_n$ are Harmonic progression then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3} \ldots \frac{1}{a_n}$ are in HP

* n^{th} term in AP $= a+(n-1)d$
* n^{th} term in HP $= \frac{1}{a+(n-1)d}$

1. $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ are in Harmonic progression the find 99^{th} term

Given series of harmonic progression is

$1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ are in Hp then $1, 3, 5, 7 \ldots$

are in Arthmetic progression

first term $= 1$

common difference $(d) = t_2 - t_1$

$d = 3 - 1$

$d = 2$

n^{th} term of Arthmetic progression

$a_n = a + (n-1)d.$

$a_{99} = a + (99-1)d$

$= a + 98d$

$= 1 + 98(2)$

$= 1 + 196$

$= 197.$

The 99^{th} term of Harmonic progression is $\frac{1}{197}$.

2. Insert 4 harmonic means between $\frac{1}{2}$ and $\frac{1}{42}$ we know that.

$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \frac{1}{a+4d}, \frac{1}{a+5d}$ are in AP.

then $a, a+d, a+2d, a+3d, a+4d, a+5d$ are in AP.

Given that

$\frac{1}{a} = \frac{1}{2}$

then $= a = 12$

and also given that

$\frac{1}{a+5d} = \frac{1}{42}$

$a + 5d = 42$

$12 + 5d = 42$

$$5d = 30$$

$$d = \cancel{30}^{6}/\cancel{5}_{1}$$

$$d = 6$$

2^{nd} term $= \dfrac{1}{a+d} \neq \dfrac{1}{12+6} = \dfrac{1}{18} = 1/18$

3^{rd} term $= \dfrac{1}{a+2d} = \dfrac{1}{12+2(6)} = \dfrac{1}{12+12} = 1/24$

4^{th} term $= \dfrac{1}{a+3d} = \dfrac{1}{12+3(6)} = \dfrac{1}{12+18} = 1/30$

5^{th} term $= \dfrac{1}{a+4d} = \dfrac{1}{12+4(6)} = \dfrac{1}{12+24} = 1/36$

$\therefore$ The four harmonic means are $1/8$, $1/24$, $1/30$, $1/36$.

1

Quantitative Analysis for Management Decisions

UNIT- IV

VECTORS AND MATRICES

Introduction:

A matrix is a way to organize the member into a rectangular rows and columns.

Definition:

- A matrix is a rectangular numbers arranged in rows and columns, and closed by [ ].
- The horizontal lines of numbers in a matrix are called Rows.
- The vertical lines of numbers in a matrix called columns.
- The numbers in a matrix or inside a matrix of numbers are called elements (or) entries.

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ 2×3

$m \times n =$ (2 rows, 3 columns)

$m =$ Rows ; $n =$ columns

Eg: Let us consider Setup of simultaneous equation.

$$x + 2y + 3z + 5t = 0$$
$$4x + 2y + 5z + 7t = 0$$
$$3x + 4y + 2z + 6t = 0$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 5 \\ 4 & 2 & 5 & 7 \\ 3 & 4 & 2 & 6 \end{bmatrix}$$ co-efficient of variables.

Where matrix is used :-

Field	Example
Mathematics	Solving complex Problems.
statistics	Data analysis
computer Science	programming, AI
Economics	Budget and sales Analysis.
Graphics	2D/3D object movements

Functions:

1. To organize data clearly:

Matrix help present information in rows and columns making last data easy handle.

2. To Solve mathematics problems:

Matrix make it easy to solve linear equation and Perform operations like addition, multiplication and invest.

3. computer science and AI:

Matrix are the bases for

1) image processing
2) Machine learning
3) Neural Networks

4) Graphics in games and apps

Business and economic :

Matrix helps in decision making and analysis, cost, profit and trend.

Importance of matrices (or) advantages :

1) To solve organizes and analysis data easily.

2) perform fast calculations using computers.

3) To solve complex problems.

4) Decision-making and analysing profits.

Limitations:

1) Difficult for last data

2) complex calculations

3) only for numerical data

Types of matrix :

1) Row matrix :

If a matrix has only one row at "n" number of columns.

Eg : $\begin{bmatrix} 5 & 8 & 3 \end{bmatrix}$

2) column matrix :

If a matrix has only one column at "n" number of rows.

Eg : $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

3) Null/zero Matrix :

Any matrix in which all elements are zero.

Eg : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4) Square Matrix:

Any matrix in which number of rows in it's is equal to number of columns.

Eg: $\begin{bmatrix} 1 & 7 & 3 \\ 2 & 6 & 4 \\ 3 & 1 & 5 \end{bmatrix}$

5) Rectangular Matrix:

A matrix with unequal number of rows and columns.

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 6 \end{bmatrix}$

6) principal Diagnol Matrix:

In a Square matrix the principle diagnol is a line of elements or numbers that goes from the top left corner, to the bottom right corner.

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ A = {1,5,9} are principal Diagnol elements.

7) Diagnol Matrix:

A Square matrix were all non-diagnol elements are zero.

Eg: $\begin{bmatrix} 6 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8) unit / Identity Matrix:

In a Square matrix if all the diagnol elements are equal to one has non-diagnol elements are equal to zero

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Eg: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

9) Scalar Matrix:

In a Square matrix were all elements are non-diagnol elements are zero and only diagnol elements are equal to each other.

Eg: $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

10) one's Matrix:

In a Square matrix were all the elements are equal to one.

Eg: $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

11) Trace of Matrix:

In a Square matrix, the sum of diagnol elements, it is denoted by T(A).

Eg: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ $T(A) = 1+5+9$

$T(A) = 15$

12) Transpose of a matrix:

In a Square matrix, 'A' is a set to be transpose of a matrix if we enter rows and columns.

It is denoted by $'A''$ (or) A^T

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

1) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ find A^T

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 6 & 5 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 5 \end{bmatrix}$$

2) If $A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$ find $A + A^T$

$$A = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & -5 \\ -4 & 3 \end{bmatrix}$$

$$A + A^T = \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+2 & -4-4 \\ -5+(-5) & 3+3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -8 \\ -10 & 6 \end{bmatrix}$$

3) 4 If $A = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix}$ Find AB^T and BA^T.

$$AB^T = \begin{bmatrix} 7 & -2 \\ -1 & 2 \\ 5 & 3 \end{bmatrix} \times \begin{bmatrix} -2 & 4 & -1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -14+2 & 28-4 & -7-0 \\ 2-2 & -4+4 & 1+0 \\ -10-3 & 20+6 & -5-0 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 24 & -7 \\ 0 & 0 & 1 \\ -13 & 26 & -5 \end{bmatrix}$$

$$BA^T = \begin{bmatrix} -2 & -1 \\ 4 & 2 \\ -1 & 0 \end{bmatrix} \times \begin{bmatrix} 7 & -1 & 5 \\ -2 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -14+2 & 2-2 & -10-3 \\ 28-4 & -4+4 & 20+6 \\ -7-0 & 1+0 & -5+0 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 0 & -13 \\ 24 & 0 & 26 \\ -7 & 1 & -5 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$ $B = \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$ then verify that (i) $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$ (iii) $(5B)^T = 5(B)^T$

Given that,

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$(A^T)^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix}$$

$$(A^T)^T = A$$

Hence proved.

(ii) $(A+B)^T$

$$A+B = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 4+4 & 7+0 \\ 2+4 & 5-2 & 8-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 8 & 7 \\ 6 & 3 & 7 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix}$$

$$B^T = \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$$A^T + B^T = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 6 & -1 \end{bmatrix}$$

5) $B^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}$

$A^T = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$

$B^T \cdot A^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$

$= \begin{bmatrix} 2+3+10 & 1-9+20 \\ -4-0+8 & -2+0+16 \end{bmatrix}$

$= \begin{bmatrix} 15 & 12 \\ 4 & 14 \end{bmatrix}$

$(AB)^T = B^T \cdot A^T$

Hence proved

13) Symmetric Matrix:-

A symmetric matrix is a square matrix that is equal to its transpose

$A = A^{-1}$

$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix}$ $A = A^T$

$A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix}$

$\therefore$ It is a symmetrix matrix

Examples:

1) If $A = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 5 \\ 6 & 5 & 9 \end{bmatrix}$ is a symmetric or not

Given that,

$A = A^T$

$$A = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 5 \\ 6 & 5 & 9 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 3 & 6 \\ 3 & 4 & 5 \\ 6 & 5 & 9 \end{bmatrix}$$

$A = A^T$

It is a Symmetric matrix

2) If $A = \begin{bmatrix} 9 & 2 & 3 \\ 2 & -1 & -8 \\ 3 & -8 & 0 \end{bmatrix}$ is a symmetric Matrix or not.

$A = A^T$

Given that,

$$A = \begin{bmatrix} 9 & 2 & 3 \\ 2 & -1 & -8 \\ 3 & -8 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 9 & 2 & 3 \\ 2 & -1 & -8 \\ 3 & -8 & 0 \end{bmatrix}$$

$A = A^T$ $\therefore$ It is a symmetrix matrix

3) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}$

Given that,

$A = A^T$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}$$

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$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 8 \end{bmatrix}$$

$A = A^T$

$\therefore$ It is a symmetric matrix.

4) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ is a symmetric matrix then find x

$A = A^T$

Given that,

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$$

$$A^T = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & x \\ 3 & 6 & 7 \end{bmatrix}$$

$\boxed{x = 6}$

6) If $A = \begin{bmatrix} 0 & 2 & 1 \\ -2 & 0 & -2 \\ -1 & x & 0 \end{bmatrix}$ is a Symmetrix matrix then find x

$A = A^T$

$$A^T = \begin{bmatrix} 0 & -2 & -1 \\ 2 & 0 & x \\ 1 & -2 & 0 \end{bmatrix}$$

$\boxed{x = -2}$

$\therefore$ Therefore;

$x = -2$

14) Non-Symmetric Matrix:

A non-symmetric Matrix is a Square if not equal to transpose of Matrix

$A \neq A^T$

Eg: $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 4 & 2 \\ 7 & 8 & 9 \end{bmatrix}$

Here $A \neq A^T$

$\therefore$ so, it is non-singular|symmetric matrix

$A^T = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 8 \\ 3 & 2 & 9 \end{bmatrix}$

1) If $A = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 5 \end{bmatrix}$ is a symmetric or not

Given that,

$A = A^T$

$A^T = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & 1 \\ 3 & 2 & 5 \end{bmatrix}$

$A \neq A^T$

$\therefore$ It is not a Symmetric matrix.

2) If $A = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & x & 7 \end{bmatrix}$ is a symmetrix matrix then find x-Written

7 15) Skew Symmetric Matrix :-

A Skew Symmetric matrix is a Square matrix were are diagonal elements are zero "A" transpose ($A^T = -A$)

Eg: $A = \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 3 \\ -4 & 3 & 0 \end{bmatrix}$

$A^T = -A$

It is a Skew Symmetric matrix

$A^T = \begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -3 \\ 4 & 3 & 0 \end{bmatrix}$

1) If $A = \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

$A^T = -A$

$= \begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 4 \\ 2 & 4 & 0 \end{bmatrix}$

$\boxed{A^T = -A}$

2) If $A = \begin{bmatrix} 0 & x & 6 \\ -4 & 0 & 2 \\ -6 & -2 & 0 \end{bmatrix}$ find x

$A = -A$

$= \begin{bmatrix} 0 & \cancel{0}4 & 6 \\ -4 & 0 & 2 \\ -6 & -2 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & -\cancel{0}4 & -6 \\ 4 & 0 & -2 \\ 6 & 2 & 0 \end{bmatrix}$

3) $\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ find x

$A = A^T$

$A^T = \begin{bmatrix} 0 & -1 & x \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{bmatrix}$

$x = -2$

$A^T = -A$

$A^T = -(-2)$

$\boxed{x = 2}$

16) Triangular Matrix :

A triangle matrix is a special matrix were all the elements either above or below the main diagonal are main elements are zero.

There are two types in triangular matrix.

1) upper triangular Matrix

2) lower triangular Matrix

1) upper Triangular Matrix :-

All elements below the main diagonal are zero

Eg : $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

upper Triangular Matrix

Lower Triangular Matrix:

All elements above the main diagonal are zero.

Eg: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

Lower triangular matrix

(17) orthogonal Matrix :-

An orthogonal matrix is a square matrix whose rows and columns are orthogonal

$AA^T = I$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$A \cdot A^T = I$

Problems

1) verify that $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is orthogonal or not:

$A \cdot A^T = I$

$A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$A \cdot A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

1) If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ show $AB \neq BA$

Given that,

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0\times0+1\times1+0\times0 & 0\times0+1\times0+0\times1 & 0\times0+1\times0+0\times0 \\ 0\times0+0\times1+1\times0 & 0\times0+0\times0+1\times0 & 0\times0+0\times0+1\times0 \\ 0\times0+0\times1+0\times0 & 0\times0+0\times0+0\times1 & 0\times0+0\times0+0\times0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0+1+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+0+1 & 0+0+0 \\ 0+0+0 & 0+0+0 & 0+0+0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0\times0+0\times0+0\times0 & 0\times1+0\times0+0\times0 & 0\times0+0\times1+0\times0 \\ 1\times0+0\times0+0\times0 & 1\times1+0\times0+0\times0 & 1\times0+0\times1+0\times0 \\ 0\times0+1\times0+0\times0 & 0\times1+1\times0+0\times0 & 0\times0+1\times1+0\times0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$AB \neq BA$

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$$= \begin{bmatrix} 0+1 & 0+0 \\ 0+0 & 1+0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow I$$

$\therefore$ It is an orthogonal matrix

2) If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ then show that $A \cdot A^T = I$

Given that,

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta \times \cos\theta + \sin\theta \times \sin\theta & \cos\theta \times -\sin\theta + \sin\theta \times \cos\theta \\ -\sin\theta \times \cos\theta + \cos\theta \times \sin\theta & -\sin\theta \times -\sin\theta + \cos\theta \times \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\sin\theta\cos\theta + \sin\theta\cos\theta \\ -\sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta & \sin^2\theta + \cos^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow I$$

3) $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \times \frac{1}{3}\begin{bmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times-1 \\ 2\times1+2\times1+2\times-2 & 2\times2+1\times1+4 & 2\times-2+1\times2+2 \\ -2\times1+2\times2+-2 & -2\times2+2\times1+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4+(-2) \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^T = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{(\sqrt{2})^2}\begin{bmatrix} 1\times1+(-1)+(-1) & 1\times1+(-1)\times1 \\ 1\times1+1\times-1 & 1\times1+1\times1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \frac{2}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

18. conjugate Matrix:-

The conjugate of matrix is found by taking the complex conjugate of each element in matrix.

It is denoted by "$\bar{A}$"

Eg: $A = \begin{bmatrix} 2+3i & 4-i \\ -1+2i & 5 \end{bmatrix}$

$\bar{A} = \begin{bmatrix} 2-3i & 4+i \\ -1-2i & 5 \end{bmatrix}$

19) Singular matrix

If the determinates of matrix is zero that the matrix it is denoted by det (A) = 0 (or) |A| = 0

Eg: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$ad-bc = 0$

$= 2\times3 - 6\times1$

$= 6-6$

$= 0$

Problems

1) $A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$

Given that,

$$A = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}$$

$\det(A) = 0$

$\det(A) = ad - bc$

$= 6(1) - 2(3)$

$\det(A) = 0$

2) $A = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$

$\det(A) = 0$

$\det(A) = ad - bc$

$= 2 \times 2 - 4 \times 1$

$= 4 - 4$

$= 0$

3) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Given that,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$

$\det |A| = 1\,((4 \times 9) - 6 \times 6) - 2\,(2 \times 9 - 6 \times 3) + 3\,(2 \times 6 - 4 \times 3)$

$= 1(36 - 36) - 2(18 - 18) + 3(12 - 12)$

$= 0 - 0 + 0$

$= 0$

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Non-Singular Matrix:-

If the determinants of the matrix is zero that the matrix is known as non-singular matrix.

It is denoted by $(A) \neq 0$

Eg:

$$A = \begin{bmatrix} 3 & 6 \\ 1 & 3 \end{bmatrix} \quad ad-bc \neq 0$$

$= 3 \times 3 - 6 \times 1$

$= 9 - 6$

$\det(A) = 3$

$\det(A) \neq 0$

1) $A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$

$\text{Det } |A| = 0$

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 2 \end{bmatrix}$$

$A = 3 \times 2 - 5 \times 4$

$= 6 - 20$

$= -14$

$\det(A) \neq 0$

B) $A = \begin{bmatrix} 2 & 3 & 1 \\ 5 & 2 & 3 \\ 2 & 4 & 1 \end{bmatrix}$

$$\text{Det } |A| = \begin{vmatrix} 2 & 3 & 1 \\ 5 & 2 & 3 \\ 2 & 4 & 1 \end{vmatrix}$$

$= 2(2-12) - 3(5-6) + 1(20-4)$

$= 2(-10) - 15 + 18 + 16$

$= -20 - 15 + 34$

$= -35 + 34$

$= -1$

Det $|A| \neq 0$

20) Addition of Matrix:-

Matrix addition is the Operating of adding any two matrices by adding the corresponding elements.

Note:

The matrices must have the same order.

Therefore, number of rows = number of columns

Eg: $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$

$$A+B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+3 & 2+1 \\ 2+2 & 3+1 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 4 & 3 \\ 4 & 4 \end{bmatrix}$$

(2) If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ $C = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ find $A+B+C$

$$A+B+C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

12 If $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ then verify that $(AB)^T = B^T \cdot A^T$

$$AB = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 10 \\ -2 & 0 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 \\ 3 & 0 \\ 10 & 16 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B^T \cdot A^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 2 + (-3\times -1) + 5\times 2 & 1\times 1 + (-3\times 3) + 5\times 4 \\ -2\times 2 + (0\times -1) + 4\times 2 & -2\times 1 + 0\times 3 + 4\times 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+10 & 1-9+20 \\ -4+8 & -2+16 \end{bmatrix} \Rightarrow \begin{bmatrix} 15 & 12 \\ 4 & 14 \end{bmatrix}$$

(ii) $(5B)^T$

$$5B = 5 \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 & 0 \\ 20 & -10 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 1-3 & 2+4 \\ 4+4 & 5-2 \\ 7+0 & 8-1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ 8 & 3 \\ 7 & 7 \end{bmatrix} \rightarrow (A+B)^T = A^T + B^T$$

Hence proved

(iii) $(5B)^T$

$$5B = 5 \begin{bmatrix} -3 & 4 & 0 \\ 4 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 & 0 \\ 20 & -10 & -5 \end{bmatrix}$$

$$(5B)^T = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

$5(B)^T$

$$B^T = \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$$= 5 \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

$(5B)^T = 5(B)^T$

Hence proved.

13 $(5B)^T = \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$

$5(B)^T$

$$B^T = \begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$$= 5\begin{bmatrix} -3 & 4 \\ 4 & -2 \\ 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -15 & 20 \\ 20 & -10 \\ 0 & -5 \end{bmatrix}$$

$$(5B)^T = 5(B)^T$$

Hence proved.

If $A = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$ then verify that $(AB)^T = B^T \cdot A^T$

Given that,

$(AB)^T = B^T \cdot A^T$

$$(AB) = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -3 & 0 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+10 & -4-0+8 \\ 1-9+20 & -2+0+16 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 4 \\ 12 & 14 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 15 & 12 \\ 4 & 14 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B^T \cdot A^T = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ -1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2+3+10 & 1-9+20 \\ -4-0+8 & -2+0+16 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 12 \\ 4 & 14 \end{bmatrix}$$

$(AB)^T = B^T \cdot A^T$

Hence proved.

If $(AB)^T = B^T \cdot A^T$ //.

If $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ -1 & 5 \end{bmatrix}$ then find $A + B^T$.

$$B^T = \begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 5 \end{bmatrix}$$

$$A + B^T = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -2+1 & 1+4 & 0-1 \\ 3+2 & 4+3 & -5+5 \end{bmatrix} = \begin{bmatrix} -1 & 5 & -1 \\ 5 & 7 & 0 \end{bmatrix}$$

21) Subtraction of Matrix:

Matrix subtraction is done by subtracting corresponding elements of two matrices.

Eg:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 & 4 \\ 6 & 7 & 4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 4 \\ 6 & 7 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-0 & 1-1 & 2-4 \\ 2-6 & 3-7 & 4-4 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0 & 0 & -2 \\ -4 & -4 & 0 \end{bmatrix}$$

2) $A = \begin{bmatrix} 9 & 7 \\ 6 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ find $A-B-C$

$$A - B - C = \begin{bmatrix} 9 & 7 \\ 6 & 5 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$$

$$A - B - C = \begin{bmatrix} 9-2-1 & 7-3-2 \\ 6-1-3 & 5-4-2 \end{bmatrix}$$

$$A - B - C = \begin{bmatrix} 6 & 2 \\ 2 & -1 \end{bmatrix}$$

$$A+B+C = \begin{bmatrix} 1+5+2 & 2+6+1 \\ 3+7+0 & 4+8+3 \end{bmatrix}$$

$$A+B+C = \begin{bmatrix} 8 & 9 \\ 10 & 15 \end{bmatrix}$$

3) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ find $A+B$

$$A+B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+2+5+3+4 \\ 4+3+5+2+6+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7+7+7 \\ 7+7+7 \end{bmatrix}$$

4) $A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 0 \\ 2 & 7 \end{bmatrix}$

$$A+B = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 \\ 2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5+4+0 \\ 1+2+3+7 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 \\ 3 & 10 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 7 & 4 \\ 3 & 10 \end{bmatrix}$$

15 3) If $A = \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 6 \\ 7 & 3 \end{bmatrix}$ find $3A-2B$

$$3A-2B = 3\begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix} - 2\begin{bmatrix} 5 & 6 \\ 7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 18 \\ 15 & 21 \end{bmatrix} - \begin{bmatrix} 10 & 12 \\ 14 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9-10 & 18-12 \\ 15-14 & 12-6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 6 \\ 1 & 15 \end{bmatrix}$$

4) $A = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ find $A^3 - 3A^2 - A - 3I = 0$

$A^3 = A^2 \times A$

$A^2 = A \times A$

$$= \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1-0+3 & -2-2-1 & 1+2+1 \\ 0+0-3 & 0+1+1 & 0-1-1 \\ 3-0+3 & -6-1-1 & 3+1+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix}$$

$A^3 = A^2 \times A$

$$= \begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & 8 & 5 \end{bmatrix} \times \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ -3 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-0+12 & -8-5-4 & 4+5+4 \\ -3+0-6 & 6+2+2 & -3-2-2 \\ 6+0+15 & -12-8-5 & 6+8+5 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix}$$

$$A^3 - 3A^2 - A - 3I$$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix} - 3\begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & -17 & 13 \\ -9 & 10 & -7 \\ 21 & -25 & 19 \end{bmatrix} - 3\begin{bmatrix} 4 & -5 & 4 \\ -3 & 2 & -2 \\ 6 & -8 & 5 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 16-12-1-3 & 17-(-15)-(2)-0 & 13-12-1-0 \\ -9-(9)-0-0 & 10-6-1-3 & -7-(-6)-(-1)-0 \\ 21-(-24)-(-1)-0 & -25-(-24)-(-1)-0 & 19-15-1-3 \end{bmatrix}.$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

22) Multiplication of Matrix :-

They are two types of multiplication of matrix.

1. Scalar Matrix
2. Matrix Matrix

1) Scalar Matrix

scalar matrix means multiplying every element of a matrix by a single number is called Scalar Matrix.

Eg:

$$\text{If } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$3A = 3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$3A = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

$$2A = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$3A - 2B = 3\begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix} - 2\begin{bmatrix} 5 & 6 \\ 7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 18 \\ 15 & 21 \end{bmatrix} - \begin{bmatrix} 10 & 12 \\ 14 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9-10 & 18-12 \\ 15-14 & 12-6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 6 \\ 1 & 15 \end{bmatrix}$$

2) Matrix Multiplication:-

Matrix Multiplication involves multiplying rows of the first matrix with columns of second Matrix.

Eg:

$$A = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix} \quad B \begin{bmatrix} 1 & 9 \\ 9 & 9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 6 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 9 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+6\times9 & 1\times9+6\times9 \\ 1\times1+4\times9 & 1\times9+4\times9 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1+54 & 9+54 \\ 1+36 & 9+30 \end{bmatrix}$$

$$AB = \begin{bmatrix} 55 & 63 \\ 37 & 45 \end{bmatrix}$$

2) $A \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ find AB

$$AB = \begin{bmatrix} a_{11} \; b_{11} + a_{12} \; b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} \; b_{11} + a_{22} \; b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$$

Power Of Matrices:-

Let A, B any matrix the multiplication of A is

$A^2 = A \times A$

$A^3 = A^2 \times A$

$A^4 = A^2 \times A^2$

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Eg:

$$A = \begin{bmatrix} -1 & 3 \\ -6 & 9 \end{bmatrix} \text{ find } A^2$$

$$= \begin{bmatrix} -1\times-1+3\times-6 & -1\times3+3\times9 \\ -6\times-1+9\times-6 & -6\times3+9\times9 \end{bmatrix}$$

$$= \begin{bmatrix} 1-18 & -3+27 \\ 6-54 & -18+81 \end{bmatrix}$$

$$= \begin{bmatrix} -17 & 24 \\ -48 & 63 \end{bmatrix} \Rightarrow A \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$A^2 = A \times A$$

$$A^2 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 2\times2+3\times1+1\times3 & 2\times3+3\times2+1\times1 & 2\times1+3\times3+1\times2 \\ 1\times2+2\times1+3\times3 & 1\times3+2\times2+3\times1 & 1\times1+2\times3+3\times2 \\ 3\times2+1\times1+2\times3 & 3\times3+1\times2+2\times1 & 3\times1+1\times3+2\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3+3 & 6+6+1 & 2+9+2 \\ 2+2+9 & 3+4+3 & 1+6+6 \\ 6+1+6 & 9+2+2 & 3+3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 13 & 13 \\ 13 & 10 & 13 \\ 13 & 13 & 10 \end{bmatrix} \times \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 20+13+39 & 30+69+13 & 10+39+26 \\ 26+10+39 & 39+20+13 & 13+30+26 \\ 26+13+30 & 39+26+10 & 13+39+30 \end{bmatrix}$$

$$= \begin{bmatrix} 72 & 69 & 75 \\ 75 & 72 & 69 \\ 69 & 75 & 72 \end{bmatrix}$$

2) If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ then Show that $A^2 - 4A - 5I = 0$

Given,

$A^2 = A \times A$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+2\times2+2\times2 & 1\times2+2\times1+2\times2 & 1\times2+2\times2+2\times1 \\ 2\times1+1\times2+2\times2 & 2\times2+1\times1+2\times2 & 2\times2+1\times2+2\times1 \\ 2\times1+2\times2+1\times2 & 2\times2+2\times1+1\times2 & 2\times2+2\times2+1\times1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$4A = 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

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$$5I = 5\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$= \begin{bmatrix}9&8&8\\8&9&8\\8&8&9\end{bmatrix}\begin{bmatrix}4&8&8\\8&4&8\\8&8&4\end{bmatrix}\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}$$

$$= \begin{bmatrix}9-4-5 & 8-8-0 & 8-8-0\\8-8-0 & 9-4-5 & 8-8-0\\8-8-0 & 8-8-0 & 9-4-5\end{bmatrix}$$

$$= \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

3) If $f(x) = x^3 - 20 - x + 8I$ where $A = \begin{bmatrix}1&1&3\\1&3&-3\\-2&4&-4\end{bmatrix}$ find $f(A)$ if $f(x) = A^3 - 20A + 8I = 0$

$A^3 = A^2 \times A$

$A^2 = A \times A$

$$= \begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}\begin{bmatrix}1&1&3\\1&3&-3\\-2&-4&-4\end{bmatrix}$$

$$= \begin{bmatrix}1+1-6 & 1+3-12 & 3-3-12\\1+3+6 & 1+9+12 & 3-9+12\\-2-4+8 & -2-12+16 & -6+12+16\end{bmatrix}$$

$$A^2 = \begin{bmatrix}-4 & -8 & -12\\10&22&6\\2&2&22\end{bmatrix}$$

$A^3 = A^2 \times A$

$$= \begin{bmatrix} -4 & -8 & -12 \\ 10 & 22 & 6 \\ 2 & 2 & 22 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -4-8+24 & -4-24+48 & -12+24+48 \\ 10+22-12 & 10+66-24 & 30-66-24 \\ 2+2-44 & 2+6-88 & 6-6-88 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 20 & 60 \\ 20 & 52 & -60 \\ -40 & -80 & -88 \end{bmatrix} + \begin{bmatrix} -20 & -20 & -60 \\ -20 & -60 & 60 \\ -40 & 80 & 80 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 12-20+8 & 20-20+0 & 60-60+0 \\ 20-20+0 & 52-60+8 & -60+60+0 \\ -40+40+0 & -80+80+0 & -88+80+8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence proved

Determinants of a Matrix

1. Find the determinant of a matrix $A = \begin{bmatrix} 3 & 5 \\ 2 & 4 \end{bmatrix}$

$\det(A) = ad - bc$

$= 3(4) - 5(2)$

$= 12 - 10$

$= 2$

$\det(A) = 2$

2) If $A = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$ find determinants of A

$$\det(A) = \begin{bmatrix} 3 & 2 & 2 \\ 1 & 3 & 1 \\ 5 & 3 & 4 \end{bmatrix}$$

$$= 3(12-3) - 2(4-5) + 2(3-15)$$

$$= 3(9) - 2(-1) + 2(-12) \Rightarrow 27 + 2 - 24$$

$$= 29 - 24 = 5$$

$\det(A) = 5$

Hence proved.

3) If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$\det(A) = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= 3(-3+4) + 3(2-0) + 4(-2-0)$$

$$= 3(1) + 3(2) + 4(-2)$$

$$= 3 + 6 - 8$$

4) If $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$$\det(A) = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$= 2(1-4) - 5(3-2) + 3(6-1)$

$= 2(-3) - 5(1) + 3(5) = -6 - 5 + 15$

$= -11 + 15 = 4$

co-factors of a matrix

The co-factors of an element in the i^{th} row and j^{th} column of 3x3 matrix is defined as it's minor is multiplied with $(-1)^{i+j}$ where i = rows and j = columns

co-factor of $A = (-1)^{i+j}$

Eg: If $A = \begin{bmatrix} 1 & 0 & 8 \\ 2 & 0 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ find co-factor of A.

co-factor of $1 = (-1)^{i+j} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$= (-1)^{1+1} [0 - 0]$

$= (-1)^2 (0)$

co-factor of $1 = 0$

cofactor of $0 = (-1)^{i+j} \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

$= (-1)^{1+2} [2 - 0]$

$= (-1)^3 (2)$

$= -1 \times 2$

co-factor of $0 = -2$

cofactor of $8 = (-1)^{1+3} \begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$

$= (-1)^4 (0)$

$= 0$

20 co-factor of 2

$= (-1)^{2+1} \begin{bmatrix} 0 & 8 \\ 0 & 1 \end{bmatrix}$

$= (-1)^3 (0)$

$= 0$

co-factor of 0 $= (-1)^{i+j} \begin{bmatrix} 1 & 8 \\ 3 & 1 \end{bmatrix}$

$= (-1)^{2+2} [1-24]$

$= (-1)^4 (-23)$

$= -23$

cofactor of 0 $= (-1)^{i+j} \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$

$= (-1)^{2+3} [0-0]$

$= 0$

co-factor of 3 $= (-1)^{3+1} \begin{bmatrix} 0 & 8 \\ 0 & 0 \end{bmatrix}$

$= (-1)^4 (0)$

$= 1 \times 0$

$= 0$

cofactor of 0 $= (-1)^{3+2} \begin{bmatrix} 1 & 8 \\ 2 & 0 \end{bmatrix}$

$= (-1)^5 [0-16]$

$= -1 \times -16$

$= 16$

co-factor of 1 $= (-1)^{3+3} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$

$= (-1)^6 (0) = 0$

cofactor of A $= \begin{bmatrix} 0 & -2 & 0 \\ 0 & -23 & 0 \\ 0 & 16 & 0 \end{bmatrix}$

2) If $A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 2 & -2 & 1 \end{bmatrix}$ find co-factor of A.

cofactor of $1 = (-1)^{1+1} \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

$= 1[3+2]$

$= 5$

co-factor of $0 = (-1)^{1+2} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$

$= (-1)(-1-2)$

$= -1(-3)$

$= 3$

co-factor of $2 = (-1)^{1+3} \begin{bmatrix} -1 & 3 \\ 2 & -2 \end{bmatrix}$

$= (-1)^4 [2-6]$

$= 1(-4)$

$= -4$

cofactor of $-1 = (-1)^{2+1} \begin{bmatrix} 0 & 2 \\ -2 & 1 \end{bmatrix}$

$= (-1)^3 [0+4]$

$= -1(4)$

$= -4$

cofactor of $3 = (-1)^{2+2} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$= (-1)^4 [1-4]$

$= 1(-3)$

$= -3$

co-factor of $1 = (-1)^{2+3} \begin{bmatrix} 1 & 0 \\ 2 & -2 \end{bmatrix}$

2)

cofactor of $1 = (-1)^{2+3}\begin{bmatrix}1 & 0\\ 2 & -2\end{bmatrix}$

$= (-1)^5[-2-0]$

$= -1[-2]$

$= 2$

cofactor of $2 = (-1)^{3+1}\begin{bmatrix}0 & 2\\ 3 & 1\end{bmatrix}$

$= (-1)^4[0-6]$

$= 1[-6]$

$= -6$

cofactor of $-2 = (-1)^{3+2}\begin{bmatrix}1 & 2\\ -1 & 1\end{bmatrix}$

$= (-1)^5[1+2]$

$= -3$

cofactor of $1 = (-1)^{3+3}\begin{bmatrix}1 & 0\\ -1 & 3\end{bmatrix}$

$= 1(3+0)$

$= 3$

cofactor of $A = \begin{bmatrix}5 & 3 & -4\\ -4 & -3 & 2\\ -6 & -3 & 3\end{bmatrix}$

3) $A = \begin{bmatrix}4 & 1 & 2\\ 0 & 3 & -1\\ 2 & 5 & 0\end{bmatrix}$

cofactor of $4 = (-1)^{1+1}\begin{bmatrix}3 & -1\\ 5 & 0\end{bmatrix}$

$= 1(0+5)$

$= 5$

cofactor of 1 = $(-1)^{1+2}\begin{bmatrix}0 & -1\\2 & 0\end{bmatrix}$

$= -1(0+2)$

$= -2$

Cofactor of 2 $= (-1)^{1+3}\begin{bmatrix}0 & 3\\2 & 5\end{bmatrix}$

$= 1(0-6)$

$= -6$

cofactor of 0 $= (-1)^{2+1}\begin{bmatrix}1 & 2\\5 & 0\end{bmatrix}$

$= -1[0-10]$

$= 10$

cofactor of 3 $= (-1)^{2+2}\begin{bmatrix}4 & 2\\2 & 0\end{bmatrix}$

$= 1(0-4)$

$= -4$

co-factor of $-1 = (-1)^{2+3}\begin{bmatrix}4 & 1\\2 & 5\end{bmatrix}$

$= -1[20-2]$

$= -1(18) = -18$

cofactor of 2 $= (-1)^{3+1}\begin{bmatrix}1 & 2\\3 & -1\end{bmatrix}$

$= 1(-1-6)$

$= 1(-7)$

$= -7$

cofactor of 5 $= (-1)^{3+2}\begin{bmatrix}4 & 2\\0 & -1\end{bmatrix}$

$= -1(-4-0) = 4$

cofactor of 0 $= (-1)^{3+3}\begin{bmatrix}4 & 1\\0 & 3\end{bmatrix}$

$= 1(12-0)$

$= 12$

cofactor of $A = \begin{bmatrix} 5 & -2 & -6 \\ 10 & -4 & -18 \\ -7 & 4 & 17 \end{bmatrix}$

Adjoint of a matrix:-

the adjoint of a square matrix in the transpose of its co-factor matrix.

consider a square matrix A, and find the co-factor of element in A. we have a co-factor matrix, then the transpose of co-factor of matrix is adjoint of A, and it is denoted by Adj(A)

$\therefore$ $Adj(A) = (\text{co-factor of } A)^T$

1) If $A = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$ find Adj(A)

Given that,

$$\begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix} = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}$$

co-factor of $(-3) = \begin{bmatrix} -1 & 1 \\ 5 & -13 \end{bmatrix}$

$= (13-5)$

$= 8$

co-factor of $(-1) = \begin{bmatrix} 1 & 1 \\ 7 & -13 \end{bmatrix}$

$= (-13-7) = 20$

$= -20$

cofactor of $5 = \begin{bmatrix} 1 & -1 \\ 7 & 5 \end{bmatrix}$

$= (5+7) = 12$

cofactor of $1 = \begin{bmatrix} -1 & 5 \\ 5 & -13 \end{bmatrix}$

$= 13 - 25$

$= -12$

cofactor of $-1 = \begin{bmatrix} -3 & 5 \\ 7 & -13 \end{bmatrix}$

$= [39 - 35]$

$= 4$

cofactor of $1 = \begin{bmatrix} -3 & -1 \\ 7 & 5 \end{bmatrix}$

$= -15 + 7$

$= -8$

cofactor of $7 = \begin{bmatrix} -1 & 5 \\ -1 & 1 \end{bmatrix}$

$= -1 + 5$

$= 4$

cofactor of $5 = \begin{bmatrix} -3 & 5 \\ 1 & 1 \end{bmatrix}$

$= -3 - 5$

$= -8$

cofactor of $-13 = \begin{bmatrix} -3 & -1 \\ 1 & -1 \end{bmatrix}$

$= 3 + 1$

$= 4$

cofactor of $A = \begin{bmatrix} 8 & 20 & 12 \\ 12 & 4 & 8 \\ 4 & 8 & 4 \end{bmatrix}$

23

$$= \begin{bmatrix} 8 & 12 & 4 \\ 20 & 4 & 8 \\ 12 & 8 & 4 \end{bmatrix}$$

2) $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

cofactor of 1 = $\begin{bmatrix} 4 & 5 \\ -6 & -7 \end{bmatrix}$

$= -28+30$

$= 2$

cofactor of 0 = $\begin{bmatrix} 3 & 5 \\ 0 & -7 \end{bmatrix}$

$= -21-0$

$= -21$

cofactor of -1 = $\begin{bmatrix} 3 & 4 \\ 0 & -6 \end{bmatrix}$

$= -18-0$

$= -18$

cofactor of 3 = $\begin{bmatrix} 0 & -1 \\ -6 & -7 \end{bmatrix}$

$= 0-6$

$= -6$

cofactor of 4 = $\begin{bmatrix} 1 & -1 \\ 0 & -7 \end{bmatrix}$

$= -7-0$

$= -7$

co-factor of 5 = $\begin{bmatrix} 1 & 0 \\ 0 & -6 \end{bmatrix}$

$= 6$

cofactor of $0 = \begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix}$

$= 0 + 4$

$= 4$

cofactor of $-6 = \begin{bmatrix} 1 & -1 \\ 3 & 5 \end{bmatrix}$

$= 5 + 3$

$= 8$

cofactor of $-7 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix}$

$= 4 - 0$

$= 4$

cofactor of $A = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$

3) If $A = \begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

co-factor of $2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$= 1 - 4$

$= -3$

co-factor of $5 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= 3 - 2 = 1$

24 cofactor of 3 = $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$= 6 - 1$

$= 5$

cofactor of 3 = $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$

$= 5 - 6$

$= -1$

cofactor of 4 = $\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

$= 2 - 3$

$= -1$

cofactor of 2 = $\begin{bmatrix} 2 & 5 \\ 1 & 2 \end{bmatrix}$

$= 4 - 5$

$= -1$

cofactor of 1 = $\begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$

$= 10 - 3$

$= 7$

cofactor of 2 = $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$

$= 4 - 9$

$= -5$

cofactor of 1 = $\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$

$= 2 - 15$

$= -13$

$$\text{Cofactor } A = \begin{bmatrix} -3 & -1 & 5 \\ 1 & -1 & 1 \\ 7 & 5 & -13 \end{bmatrix}$$

$$Adj(A) = \begin{bmatrix} -3 & 1 & 7 \\ -1 & -1 & 5 \\ 5 & 1 & -13 \end{bmatrix}$$

Inverse of a matrix (or) Inverse matrix:

Let A be a Square matrix with 'n' rows and columns, If there is an m×n matrix, B such that AB = I and BA = I then A & B are inverse of one another. The inverse of matrix 'A' is denoted by A^{-1}.

$$A^{-1} = \frac{1}{\det(A)} \cdot Adj(A)$$

1) If $A = \begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix}$ for A^{-1}

$$\det A = 3(-3+2) - 1(2+1) + 2(4+3)$$

$$= 3(-1) - 1(3) + 2(7)$$

$$= -3 - 3 + 14$$

$$= -6 + 14$$

$$\det(A) = 8$$

$$\text{Cofactor of } 3 = \begin{bmatrix} -3 & -1 \\ 2 & 1 \end{bmatrix}$$

$$= -3 + 2$$

$$= -1$$

25

cofactor of $1 = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$

$= 2+1$

$= 3$

cofactor of $2 = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix}$

$= 4+3$

$= 7$

co-factor of $2 = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

$= 1-4$

$= -3$

co-factor of $-3 = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$

$= 3-2$

$= 1$

cofactor of $-1 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$

$= 6-1$

$= 5$

cofactor of $1 = \begin{bmatrix} 1 & 2 \\ -3 & -1 \end{bmatrix}$

$= -1+6$

$= 5$

cofactor of $2 = \begin{bmatrix} 3 & 2 \\ 2 & -1 \end{bmatrix}$

$= -3-4$

$= -7$

Cofactor of $1 = \begin{bmatrix} 3 & 1 \\ 2 & -3 \end{bmatrix}$

$= -9-2 = -11$

cofactor of A = $\begin{bmatrix} -1 & -3 & 7 \\ 3 & 1 & -5 \\ 5 & 7 & -11 \end{bmatrix}$

Adj(A) = $\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$

$A^{-1} = \frac{1}{8}\begin{bmatrix} -1 & 3 & 5 \\ -3 & 1 & 7 \\ 7 & -5 & -11 \end{bmatrix}$

2) If A = $\begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 1 & 0 & 8 \end{bmatrix}$

det (A) = 1 (16-0) - 0 (32-0) + 0

= 16

cofactor of 1 = $\begin{bmatrix} 2 & 0 \\ 0 & 8 \end{bmatrix}$

= 16 - 0

= 16

cofactor of 0 = $\begin{bmatrix} 4 & 0 \\ 1 & 8 \end{bmatrix}$

= 32 - 0

= 32

cofactor of 0 = $\begin{bmatrix} 4 & 2 \\ 1 & 0 \end{bmatrix}$

= 4 - 2

= 2

26

cofactor of 4 = $\begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix}$

$= 0 - 0$

$= 0$

cofactor of 2 = $\begin{bmatrix} 1 & 0 \\ 1 & 8 \end{bmatrix}$

$= 8 - 0$

$= 8$

cofactor of 0 = $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$= 0 - 0$

$= 0$

cofactor of 1 = $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$

$= 0 - 0$

$= 0$

cofactor of 0 = $\begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix}$

$= 0 - 0$

$= 0$

cofactor of 8 = $\begin{bmatrix} 1 & 0 \\ 4 & 2 \end{bmatrix}$

$= 2 - 0$

$= 2$

Cofactor of A = $\begin{bmatrix} 16 & -32 & -2 \\ 0 & 8 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Adj(A) = $\begin{bmatrix} 16 & 0 & 0 \\ -32 & 8 & 0 \\ -2 & 0 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{16}\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1/2 & 0 \\ -1/8 & 0 & 1/8 \end{bmatrix}$$

3) If $A = \begin{bmatrix} -3 & 2 & -1 \\ 4 & 3 & 5 \\ 6 & 5 & -6 \end{bmatrix}$ find A^{-1}

$\text{Det}(A) = -3(-18-25) - 2(-24-30) - 1(20-18)$

$= -3(-43) - 2(-54) - 1(2)$

$= 129 + 108 - 2$

$= 237 - 2$

$= 235$

cofactor of $-3 = \begin{bmatrix} 3 & 5 \\ 5 & -6 \end{bmatrix}$

$= -18 - 25$

$= -43$

cofactor of $2 = \begin{bmatrix} 4 & 5 \\ 6 & -6 \end{bmatrix}$

$= -24 - 30$

$= -54$

cofactor of $-1 = \begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$

$= 20 - 18$

$= 2$

cofactor of $4 = \begin{bmatrix} 2 & -1 \\ 5 & -6 \end{bmatrix}$

$= -12 + 5$

$= -7$

27

co factor of 3 = $\begin{bmatrix} -3 & -1 \\ 6 & -6 \end{bmatrix}$

$= 18+6$

$= 24$

cofactor of 5 = $\begin{bmatrix} -3 & 2 \\ 6 & 5 \end{bmatrix}$

$= -15-12$

$= -27$

co factor of 6 = $\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}$

$= 10+3$

$= 13$

co factor of 5 = $\begin{bmatrix} -3 & -1 \\ 4 & 5 \end{bmatrix}$

$= -15+4$

$= -11$

co factor of -6 = $\begin{bmatrix} -3 & 2 \\ 4 & 3 \end{bmatrix}$

$= -9-8$

$= -17$

co-factor of A =

$$\begin{bmatrix} -43 & 54 & 2 \\ 7 & 24 & 27 \\ 13 & 11 & -17 \end{bmatrix}$$

Adj(A) => $\begin{bmatrix} -43 & 7 & 13 \\ 54 & 24 & 11 \\ 2 & 27 & -17 \end{bmatrix}$

$$A^{-1} = \frac{1}{235}\begin{bmatrix} -43 & 7 & 13 \\ 54 & 24 & 11 \\ 2 & 27 & -17 \end{bmatrix}$$

Rank of a matrix : $\rho(A)$

The rank of a matrix in the maximum number of linearly independent Rows and columns in the matrix conditions.

2x2 Matrix :

1. The determinant value is 0, then the rank is 1.
2. The determinant value is not equal to 0, then the rank is 2.

3x3 Matrix:

1. The determinant value is 0, if we remove any row and column of given matrix convert into 2x2 matrix, then $\rho(A) = 2$.
2. The determinant value is not equals to zero, then the rank of the matrix is '3'.

1) If $A = \begin{bmatrix} 0 & 8 \\ 0 & 9 \end{bmatrix}$ find Rank

$\det(A) = (0-0)$

$\det = 0$

$\therefore$ It is a Singular matrix

$\therefore \rho(A) = 1$

2) If $A = \begin{bmatrix} 6 & 3 \\ 2 & 1 \end{bmatrix}$

$\text{Det}(A) = 6-6$

$= 0$

$\rho(A) = 1$ $\therefore$ It is a singular matrix

28 3) If $A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$

$Det(A)=0-0$

$\therefore$ It is a singular Matrix

$\rho(A)=1$

4) If $A=\begin{bmatrix}5 & 6\\ 8 & 9\end{bmatrix}$

$Det\ A=(45-48)$

$=-3$

$det\ (A)\neq 0$

It is a non-singular matrix

$\rho(A)=2$

5) If $A=\begin{bmatrix}9 & 9\\ 8 & 0\end{bmatrix}$

$Det\ A=0-64$

$=-64$

$\therefore$ It is a non-singular matrix

$\therefore \rho(A)=2$

6) If $A=\begin{bmatrix}9 & 3 & 4\\ 5 & 0 & 2\\ 4 & 3 & 2\end{bmatrix}$ find $\rho(A)$

$det\ (A)=9(0-6)-3(10-8)+4(15-0$

$=9(-6)-3(2)+4(15)$

$=-54-6+60$

$= -60+60$

$= 0$

$\det(A) = 0$

$\therefore$ It is a singular matrix

By eliminating 1^{st} row and 1^{st} column

Let $B = \begin{bmatrix} 0 & 2 \\ 3 & 2 \end{bmatrix}$

$\det B = (0-6)$

$\det B = -6$

$\det B \neq 0$

$\therefore \rho(A) = 2$

7) If $A = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 4 & 3 \\ 5 & 2 & 1 \end{bmatrix}$ find $\rho(A)$

$\det(A) = 1(4-6) - 4(2-15) + 5(4-20)$

$= 1(-2) - 4(-13) + 5(-16)$

$= -2 + 52 - 80$

$= -30$

It is non-singular matrix

By eliminating

$A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$

$= 4-6 = -2$

$\therefore$ It is non-singular

$\rho(A) = 3$

29 ## Matrix inversion Method:

1) $\begin{bmatrix} x+y-z = 0 \\ 2x-3y+z = 1 \\ 2x+y+2z = 7 \end{bmatrix}$

Given that,

$$\begin{bmatrix} x+y-z = 0 \\ 2x-3y+z = 1 \\ 2x+y+2z = 7 \end{bmatrix}$$

convert given equation into matrix form

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & 1 & 2 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{\det A} \text{Adj}(A)$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$\det A = 1(-6-1) - 1(4-2) - 1(2+6)$$

$$= -7-2-8$$

$$= -17$$

To find co-factor

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

cofactor of 1 = $\begin{vmatrix} -3 & 1 \\ 1 & 2 \end{vmatrix}$

$= -6 - 1$

$= -7$

co-factor of 1 = $\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix}$

$= 4 - 2$

$= 2$

cofactor of -1 = $\begin{vmatrix} 2 & -3 \\ 2 & 1 \end{vmatrix}$

$= 2 + 6$

$= 8$

cofactor of 2 = $\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}$

$= 2 + 1$

$= 3$

cofactor of -3 = $\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix}$

$= 2 + 2 = 4$

cofactor of 1 = $\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}$

$= 1 - 2$

$= -1$

cofactor of 2 = $\begin{vmatrix} 1 & -1 \\ -3 & 1 \end{vmatrix}$

$= 1 - 3 = -2$

cofactor of 1 = $\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}$

$= 1 + 2$

$= 3$

cofactor of 2 = $\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix}$

$= -3 - 2 = -5$

30 co-factor of $A = \begin{bmatrix} -7 & -2 & 8 \\ -3 & 4 & 1 \\ -2 & -3 & -5 \end{bmatrix}$

$$\mathrm{Adj}(A) = \begin{bmatrix} -7 & -3 & -2 \\ -2 & 4 & -3 \\ 8 & 1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-17} \begin{bmatrix} -7 & -3 & -2 \\ -2 & 4 & -3 \\ 8 & 1 & 5 \end{bmatrix}$$

$x = A^{-1}B$

$$= \frac{1}{-17} \begin{bmatrix} -7 & -3 & -2 \\ -2 & 4 & -3 \\ 8 & 1 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 7 \end{bmatrix}$$

$$= \frac{1}{-17} \begin{bmatrix} 0 + (-3) + (-14) \\ 0 + 4 - 21 \\ 0 + 1 - 35 \end{bmatrix}$$

$$= \frac{1}{-17} \begin{bmatrix} -17 \\ -17 \\ -34 \end{bmatrix}$$

$$= 1 \begin{bmatrix} -17/-17 \\ -17/-17 \\ -17/-34 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$x = 1 ; y = 1 ; z = 2$

2) $\begin{bmatrix} x+y+z=1 \\ 2x+2y+3z=6 \\ x+4y+9z=3 \end{bmatrix}$

Given that,

$$\begin{bmatrix} x+y+z=1 \\ 2x+2y+3z=6 \\ x+4y+9z=3 \end{bmatrix}$$

conversion;

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$$

$X = A^{-1}B$

$$A^{-1} = \frac{1}{\det A} \cdot (\text{Adj} A)$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

$= 1(18-12) - 1(18-3) + 1(8-2)$

$= 6-15+6$

$= 12-15$

$= -3$

To find co-factor

co-factor of $1 = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}$

$= 18-12 = 6$

3) co-factor of $1 = \begin{vmatrix} 2 & 3 \\ 1 & 9 \end{vmatrix}$

$= 18 - 3$

$= 15$

co factor of $1 = \begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}$

$= 8 - 2$

$= 6$

co factor of $2 = \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix}$

$= 9 - 4$

$= 5$

co factor of $2 = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix}$

$= 9 - 1$

$= 8$

co factor of $3 = \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix}$

$= 4 - 1$

$= 3$

co-factor of $1 = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

$= 3 - 2$

$= 1$

co factor of $4 = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$

$= 3 - 2$

$= 1$

co factor of $9 = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}$

$= 2 - 2$

$= 0$

Co-factor of $A = \begin{bmatrix} 6 & -15 & 6 \\ -5 & 8 & -3 \\ 1 & -1 & 0 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$

$A^{-1} = \frac{1}{\det A}(Adj\, A)$

$= \frac{1}{-3}\begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}$

$x = A^{-1}B$

$= \frac{1}{-3}\begin{bmatrix} 6 & -5 & 1 \\ -15 & 8 & -1 \\ 6 & -3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 6 \\ 3 \end{bmatrix}$

$= \frac{1}{-3}\begin{bmatrix} 6-30+3 \\ -15+48-3 \\ 6-18+0 \end{bmatrix}$

$= \frac{1}{-3}\begin{bmatrix} -21 \\ 30 \\ -12 \end{bmatrix}$

$= 1 \begin{bmatrix} \frac{-21}{-3} \\ \frac{30}{-3} \\ \frac{-12}{-3} \end{bmatrix}$

$x = \begin{bmatrix} 7 \\ -10 \\ 4 \end{bmatrix}$ $x = 7;\ y = -10;\ z = 4$

3) 32 $\begin{bmatrix} x+y+z=6 \\ 2x+3y+5z=4 \\ 4x+10y+5z=10 \end{bmatrix}$

Given that,

$$\begin{bmatrix} x+y+z=6 \\ 2x+3y+5z=4 \\ 4x+10y+5z=10 \end{bmatrix}$$

conversion;

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 0 & 5 \end{bmatrix}$$

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 0 & 5 \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B=\begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$$

$X=A^{-1}B$

$$A^{-1}=\frac{1}{\det A}\cdot(Adj(A))$$

$$\det A=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 0 & 5 \end{bmatrix}$$

$=1(15-0)-1(10-20)+1(0-12)$

$=15-1(-10)+1(-12)$

$=15+10-12$

$=25-12$

$=13$

cofactor of $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 0 & 6 \end{bmatrix}$

cofactor of $1 = 1(15-0)$

$= 15$

cofactor of $1 = 1(10-20)$

$= -10$

co-factor of $1 = 0-12$

$= -12$

co-factor of $2 = 5-0$

$= 5$

co-factor of $3 = 5-4$

$= 1$

co-factor of $5 = 0-4$

$= -4$

co-factor of $4 = 5-3$

$= 2$

co-factor of $0 = 5-2$

$= 3$

cofactor of $5 = 3-2$

$= 1$

cofactor of $A = \begin{bmatrix} 15 & 10 & -12 \\ -5 & 1 & 4 \\ 2 & -3 & 1 \end{bmatrix}$

$Adj(A) = \begin{bmatrix} 15 & -5 & 2 \\ 10 & 1 & -3 \\ -12 & 4 & 1 \end{bmatrix}$

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$$A^{-1}_{of} = \frac{1}{\det A}(\text{Adj } A)$$

$$= \frac{1}{13}\begin{bmatrix} 15 & -5 & 2 \\ 10 & 1 & -3 \\ -12 & 4 & 1 \end{bmatrix}$$

$$x = A^{-1}B$$

$$= \frac{1}{13}\begin{bmatrix} 15 & -5 & 2 \\ 10 & 1 & -3 \\ -12 & 4 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 4 \\ 10 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 90-20+20 \\ 60+4-30 \\ -72+16+10 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} 90 \\ 34 \\ -46 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{90}{13} \\ \frac{34}{13} \\ -\frac{46}{13} \end{bmatrix}$$

Cramer's Rule:

1. $$\begin{bmatrix} 2x-y+3z = 9 \\ x+y+z = 6 \\ x-y+z = 2 \end{bmatrix}$$

Given that,

$$\begin{bmatrix} 2x-y+3z = 9 \\ x+y+z = 6 \\ x-y+z = 2 \end{bmatrix}$$

conversion;

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 9 \\ 6 \\ 2 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\text{Det } A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$= 2(1+1)+1(1-1)+3(-1-1)$

$= 2(2)+1(0)+3(-2)$

$= 4+0-6$

$\Delta = -2$

$$\Delta_1 = \begin{bmatrix} 9 & -1 & 3 \\ 6 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$$

$= 9(1+1)+1(6-2)+3(-6-2)$

$= 9(2)+1(4)+3(-8)$

$= 18+4-24$

$= 22-24$

$\Delta_1 = -2$

$$\Delta_2 = \begin{bmatrix} 2 & 9 & 3 \\ 1 & 6 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$= 2(6-2)-9(1-1)+3(2-6)$

$= 2(4)-9(0)+3(-4)$

$= 8-0-12 = -4$

34

$$\Delta_3 = \begin{bmatrix} 2 & -1 & 9 \\ 1 & 1 & 6 \\ 1 & -1 & 2 \end{bmatrix}$$

$= 2(2+6)+1(2-6)+9(-1-1)$

$= 2(8)+1(-4)+9(-2)$

$= 16-4-18$

$= 16-22$

$\Delta_3 = -6$

$x = \frac{\Delta_1}{\Delta}$

$= \frac{-2}{-2} = 1$

$y = \frac{\Delta_2}{\Delta}$

$= \frac{-4}{-2}$

$= 2$

$z = \frac{\Delta_3}{\Delta} = \frac{-6}{-2}$

$= 3$

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

2)

$$\begin{bmatrix} 2x+3y-5z=1 \\ x+y-z=2 \\ 0+2y+z=8 \end{bmatrix}$$

Given that,

$$\begin{bmatrix} 2x + 3y - 5z = 1 \\ x + y - z = 2 \\ 0 + 2y + z = 8 \end{bmatrix}$$

conversion;

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 1 & 1 & -1 \\ 0 & 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}$$

$$\Delta = 2(1+2) - 3(1+0) - 5(2-0)$$

$$= 2(3) - 3(1) - 5(2)$$

$$= 6 - 3 - 10$$

$$= 6 - 13$$

$$= -7$$

$$\Delta_1 = \begin{bmatrix} 1 & 3 & -5 \\ 2 & 1 & -1 \\ 8 & 2 & 1 \end{bmatrix}$$

$$= 1(1+2) - 3(2+8) - 5(4-8)$$

$$= 1(3) - 3(10) - 5(-4)$$

$$= 3 - 30 + 20$$

$$= -27 + 20$$

$$= -7$$

$$\Delta_2 = \begin{bmatrix} 2 & 1 & -5 \\ 1 & 2 & -1 \\ 0 & 8 & 1 \end{bmatrix}$$

$$= 2(2+8) - 1(1+0) - 5(8-0)$$

35

$= 2(10) - 1 - 40$

$= 20 - 40 - 1$

$= -21$

$$\Delta_3 = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 8 \end{bmatrix}$$

$= 2(8-4) - 3(8-0) + 1(2-0)$

$= 2(4) - 3(8) + 1(2)$

$= 8 - 24 + 2$

$= 8 - 24 + 2$

$= -14$

$x = \frac{\Delta_1}{\Delta} = \frac{-7}{-7}$

$= 1$

$y = \frac{\Delta_2}{\Delta} = \frac{-21}{-7}$

$= 3$

$z = \frac{\Delta_3}{\Delta} = \frac{-18}{-7}$

$= \frac{14}{7} = 2$

$\therefore x = 1;\ y = 3;\ z = 2$

3) $x + 2y + 3z = 5$

$3x + y - 3z = 4$

$-3x + 4y + 7z = -7$

Given that,

$x + 2y + 3z = 5$

$3x + y - 3z = 4$

$-3x + 4y + 7z = -7$

Conversion;

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & -3 \\ -3 & 4 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 5 \\ 4 \\ -7 \end{bmatrix}$$

$$\Delta = 1(7+12) - 2(21-9) + 3(12+3)$$

$$= 1(19) - 2(12) + 3(15)$$

$$= 19 - 24 + 45$$

$$= 64 - 24$$

$$= 40$$

$$\Delta_1 = \begin{bmatrix} 5 & 2 & 3 \\ 4 & 1 & -3 \\ -7 & 4 & 7 \end{bmatrix}$$

$$= 5(7+12) - 2(28-21) + 3(16+7)$$

$$= 5(19) - 2(7) + 3(23)$$

$$= 95 - 14 + 69$$

$$= 150$$

$$\Delta_2 = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 4 & -3 \\ -3 & -7 & 7 \end{bmatrix}$$

$$= 1(28-21) - 5(21-9) + 3(-21+12)$$

$$= 1(7) - 5(12) + 3(-9)$$

$$= 7 - 60 - 27$$

$$= 7 - 87$$

$$= -80$$

$$\Delta_3 = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ -3 & 4 & -7 \end{bmatrix}$$

36

$= 1(-7-16) - 2(-21+12) + 5(12+3)$

$= -23 + 18 + 75$

$= -23 + 93$

$= 70$

$x = \frac{\Delta_1}{\Delta} = \frac{150}{40}$

$y = \frac{\Delta_2}{\Delta} = \frac{-80}{40}$

$z = \frac{\Delta_3}{\Delta} = \frac{70}{40}$

Vectors :-

As vector is an ordered array of numbers written in a Single row (or) single column. Each number is known as a component (or) element of the vector.

Ex : $\vec{A} \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ column vector = (3 rows, 1 column) ($\rightarrow$ = Line segment)

Applications :-

Area	Application
Marketing	Represent sales, Growth and customer response in multiple regions.
Protection Mangement	Represent quantities of resources used in multiple products.
Economics	Represent output, cost (or) demand for various industries.

Matrix applications in management :-

Matrix/Matrices are applied in management for tasks such as financial analysis, resource allocation, and sales forecasting by using linear algebra to solve complex problems and handle large datasets. Additionally, matrices are used to Structure organizations, with report to multiple Supervisors. Decision-making tools, like risk and decision matrices, also use matrix principles to help evaluate options and prioritize tasks.

Quantitative and analytical applications

Financial analysis:-

Used for managing investment portfolios, analyzing market trends, and calculating total spending from purchase quantities and prices.

Resource Allocation:-

Optimizes the distribution of resources in supply chains and operations by representing production capacities and resource requirements in a matrix.

Sales Forecasting:-

Analyzes historical data to predict future sales patterns.

Risk Management:-

Help assess and manage risks by representing the like lihood and impact of different risks to prioritize mitigation strategies

Linear programming:-

Used to solve systems of different risks to prioritize mitigation strategies.

Used to solve systems of linear equations to determine optimal outcomes, such as the most profitable mix of ingredients in a product.

Project Management:-

Manages project schedules and resources by rep-resenting task dependencies and optimizing time lines

38 Data organization:-

Represents large quantities of data, such as quarterly sales or staff numbers, in a compact and easily calculable format.

Organizational and Structural Applications:-

Matrix Management:-

A structure where employees report to more than one manager (e.g; functional manager and a project manager.

Decision-Making Tools:-

creates decision matrices to a comparision of multiple Solutions, based on several quantitative measures, helping to make composite decisions.

Organizational Structures:-

can be used to create flexible Organizational structures that combine functional departments with project teams, allowing for cross-functional collabration.

UNIT 5

GAME THEORY

Game theory is the study of strategic interactions where the outcome for each participant, or "player," depends on the choices of all players. Its scope includes fields like economics, political science, and biology, analyzing situations of both cooperation and conflict among rational decision-makers. Its significance lies in providing a framework to model and predict behavior in interdependent situations, leading to optimal decision-making strategies in areas from business to international relations.

Meaning

- Game theory studies strategic decision-making where an individual's outcome is a function of their own choices and those of others.
- It assumes players are rational, make choices to maximize their own outcomes, and consider how others' strategies will affect them.
- Key terms include "players" (individuals, companies, nations), "strategies" (actions they can take), and "payoffs" (the rewards or outcomes).

Scope

- **Economics**: Used to analyze price competition, market strategy, and the behavior of firms.
- **Political Science**: Applies to international relations, helping nations decide when to cooperate or compete.
- **Biology**: Used to study evolutionary strategies and behavior in species.
- **Artificial Intelligence**: Provides a formal structure for analyzing strategic interactions in multi-agent systems.
- **Project Management**: Helps understand differing motivations within a project team, such as a manager's goal versus a worker's.
- **Other Applications**: Can be applied to diverse situations like auctions, partnerships, and even simple games like rock-paper-scissors.

Significance

- **Predictive Power**: It offers a way to analyze and predict outcomes in complex, interdependent situations.
- **Formal Language**: It provides a common language and mathematical framework for structuring and analyzing strategic scenarios.
- **Optimal Strategy**: It helps individuals, companies, or nations find optimal strategies in competitive or cooperative situations.
- **Understanding Interdependence**: It highlights the core concept that one's success often depends on the choices of others, forcing a consideration of "what if" scenarios based on others' thinking.
- **Foundation of Modern Analysis**: Many modern economic models and fields like evolutionary biology rely heavily on game theory to understand strategic behavior.

Characteristics

- **Players:** The decision-makers in the game (individuals, companies, nations, etc.).
- **Strategies:** The possible courses of action a player can take.
- **Payoffs:** The outcomes or results (gains or losses) a player receives for each combination of strategies chosen by all players.
- **Rationality:** Players are assumed to be rational and want to maximize their own gains while minimizing losses.

Theory

- **Interdependence:** The core idea is that one player's actions have consequences for other players.
- **Rational Choice:** Game theory provides a framework for choosing the best strategy by considering what an opponent might do.
- **Classification:** Games can be categorized based on different factors, such as:
  - **Zero-sum vs. Non-zero-sum:** In zero-sum games, one player's gain is another's loss, while non-zero-sum allows for both to win or lose.
  - **Cooperative vs. Non-cooperative:** Players can either form binding agreements (cooperative) or not (non-cooperative).
  - **Sequential vs. Simultaneous:** Players either move in a sequence, or at the same time.

Rules

- **Definition:** A set of rules that defines the game and the constraints on player behavior.
- **Structure:** The rules determine how actions are taken, the order of play, and the information available to players at each step.
- **Game Structure:** Rules can be physical, ecological, or social and can sometimes be altered by powerful actors or participants.

Business applications

- **Pricing:** Businesses use game theory to model how changes in price by one company affect its competitors and market share. A classic example is the "prisoner's dilemma," which shows how firms might be tempted to lower prices, but if both do, they both end up with lower profits.
- **Advertising:** Game theory can help analyze the effectiveness of advertising by modeling the interactions between advertisers and consumers, guiding businesses on how to create more impactful campaigns to maximize their return on investment.
- **Competitive analysis:** It is used to examine the competitive landscape of an industry by modeling interactions among firms to understand competitor behavior and develop tactics to gain an advantage.
- **Strategic alliances:** Businesses can use game theory to assess the pros and cons of forming alliances with other firms by modeling how different alliance structures might influence market share and profitability.
- **Mergers and acquisitions:** Game theory can analyze the strategic interactions between companies involved in a merger or acquisition to help identify the most favorable outcomes and aid in decision-making.

Key concepts in game theory for business

- **Rationality:** The assumption that players (businesses) act to maximize their own rewards, such as profit.
- **Best response:** The strategy that yields the highest payoff for a player, given the other players' strategies.
- **Nash equilibrium:** A state where no player can improve their outcome by unilaterally changing their strategy, assuming the other players' strategies remain the same.
- **Dominant strategy:** A strategy that is the best choice for a player regardless of what the other players do.

4. Game theory

Game :- A compitetive situation is called a game. If it has the following properties..

1. There are finite no. of players.
2. Each player has a finite no. of strategies.
3. Every game results in an outcome.

Terminology:-

1. No. of players :- If a game involves only 2 players then it is called 2 person game. If no. of players 1 or more than 2 then the game is known as n person game.
2. Two person zero sum game :- A game with only two players is called 2 persom zero sum game, if 1 players game is equal is to the loss of other players, then the total sum is zero.
3. strategy :- The list of all possible actions that the player will take for every pay off is called strategy.

*. Optimal strategy :- The particular strategy by which a player optimize its gain or losses without knowing the competitors strategies is called optimal strategy. They are

a. Pure strategy :- It is the decision rule which is always used by the players to select particular strategy, each player known all the strategies in advance.

b. Mixed strategies :- The course of action that are to be selected on a particular occasion with same fixed probabilities are called mixed strategy.

4. Value of the game :- The expected outcome for playing when player flow their optimal strategy is called value of the game.

Pay off matrix :- A quantitative method of satisfaction player gets at the end the playing is called pay off game. The form of a matrix by these values is called pay off matrix.

$$\begin{array}{c c} & \text{Player B} \\ & \begin{array}{cccc} B_1 & B_2 & \cdots & B_n \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \end{array}$$

Maxmin principle :- For player A the minimum value in each row represents least gain in the matrix by row minima then select the largest gain among the row minimum value is called maxmin principle.

Minmax principle :- For player B is assumed to be the looser the maximum value in each column in the matrix by column maximum then select minimum lose among the column maximum values is called minmax principle.

Saddle point :- If maxmin value is equal to minmax value then the game is said to be saddle point.

Method-1 :- finding saddle point.

1. Select the rowminima & enclosed it in a rectangle.
2. Select the column maxima and enclosed with in a circle.
3. Find out the element which is some in the circle as well as rectangle.
4. The elements of these represents value of the game and is called saddle point.

① Determine the optimal strategy and also value of the game to the following pay off.

$$\begin{array}{c c} & \text{Player B} \\ & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \text{Player A} \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -1 & 2 & -2 \\ 6 & 4 & -6 \end{bmatrix} \end{array}$$

$$\begin{array}{c c c} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} & \text{row min} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -1 & 2 & \boxed{-2} \\ 6 & 4 & \boxed{-6} \end{bmatrix} & \begin{array}{c} -2 \\ -6 \end{array} \\ \text{col max} & \begin{array}{ccc} 6 & 4 & -2 \end{array} & \end{array}$$

Min max = $\{6, 4, -2\} = -2$

Maxmin = max $\{-2, -6\} = -2$

∴ saddle point exist.

∴ The pure strategy is = $\{A_1 B_3\}$

Value of the Game = -2 $\left\{\begin{matrix} +ve \text{ player } A \\ -ve \text{ player } B \end{matrix}\right\}$

player B won the game.

(5) solve the following game using maxmin & minmax principle whose pay off matrix is given below.

(a)

$$\begin{array}{c|c} & \text{player A} \\ \text{player B} & \begin{array}{cc} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix} \end{array} \end{array}$$

$$\begin{array}{cc} \begin{bmatrix} \boxed{1} & 7 & 3 & 4 \\ 5 & 6 & \boxed{4} & 5 \\ 7 & 2 & \boxed{0} & 3 \end{bmatrix} & \begin{array}{c} \text{row min} \\ 1 \\ 4 \\ 0 \end{array} \\ \text{colmax } 7 \quad 7 \quad 4 \quad 5 & \end{array}$$

minmax = min{7, 7, 4, 5} = 4

maxmin = max{1, 4, 0} = 4

∴ saddle point exist.

∴ pure strategy is = $\{A_2 B_3\}$

Value of game = 4

player A won the game.

(b)

$$\begin{array}{c|c} & \text{player A} \\ \text{player B} & \begin{array}{cc} & \begin{array}{ccc} B_1 & B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} & \begin{bmatrix} 2 & 4 & 5 \\ 10 & 7 & 8 \\ 4 & 6 & 6 \end{bmatrix} \end{array} \end{array}$$

$$\begin{array}{cc} \begin{bmatrix} \boxed{2} & 4 & 5 \\ 10 & \boxed{7} & 8 \\ \boxed{4} & 6 & 6 \end{bmatrix} & \begin{array}{c} \text{row min} \\ 2 \\ 7 \\ 4 \end{array} \\ \text{colmax } 10 \quad 7 \quad 8 & \end{array}$$

minmax = min{10, 7, 8} = 7

maxmin = max{2, 7, 4} = 7

∴ saddle point exist

∴ pure strategy is = $\{A_2 B_2\}$

value of game = 7

player A won the game.

Mixed strategy without saddle point:-

In certain cases the saddle point does not exist to find the solution of such case, both the players must determine the optimal mixed strategy. A mixed strategy game can be solved by the following methods:-

1. Algebraic method. (2×2)

2. Dominance method, $(m \times n)$

3. Graphical method $(2\times n, n\times 2)$

$$\begin{array}{c|cc} & \multicolumn{2}{c}{P.B} \\ & B_1 & B_2 \\ A_1 & a & b \\ A_2 & c & d \end{array}$$

Algebraic method :- consider the payoff matrix P.A $\begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

In this method minmax $\neq$ maxmin. then consider the probabilities $P_1 P_2$, q_1, q_2 i.e., $P_1 = \dfrac{d-c}{(a+d)-(b+c)}$, $P_2 = 1-P_1$ &

$$q_1 = \frac{d-b}{(a+d)-(b+c)}, \quad q_2 = 1-q_1$$

$\therefore$ The mixed strategies for the players A & B are

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 & P_2 \end{bmatrix} \quad \& \quad S_B = \begin{bmatrix} B_1 & q_1 \\ B_2 & q_2 \end{bmatrix}$$

$\therefore$ The value the game "V" $= \dfrac{ad-bc}{(a+d)-(b+c)}$

① For the game with payoff matrix determine the optimal strategy and value of the game.

$$\begin{array}{cc} & \begin{array}{cc} B_1 & B_2 \end{array} \rightarrow \text{Player II} \\ \text{player I} & \begin{array}{c} A_1 \\ A_2 \end{array}\begin{bmatrix} 5 & 1 \\ 3 & 4 \end{bmatrix} \end{array}$$

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{rowmin} \\ A_1 & 5 & 1 & 1 \\ A_2 & 3 & 4 & 3 \\ \text{colmax} & 5 & 4 & \end{array}$$

minmax $= \min\{5, 4\} = 4$

maxmin $= \max\{1, 3\} = 3$

$\therefore$ minmax $\neq$ maxmin

$4 \neq 3$.

$\therefore$ The saddle point does not exist

$\therefore$ Consider the probabilities P_1, P_2, q_1, q_2.

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{4-3}{(5+4)-(1+3)}$$

$$= \frac{1}{9-4} = \frac{1}{5}$$

$$P_2 = 1-P_1 = 1-\frac{1}{5} = \frac{4}{5}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{4-1}{(5+4)-(1+3)} = \frac{3}{5}$$

$$q_2 = 1-q_1 = 1-\frac{3}{5} = \frac{2}{5}$$

∴ The mixed strategies for both the players.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 1/5 & P_2 = 4/5 \end{bmatrix} \quad \& \quad S_B = \begin{bmatrix} B_1 & q_1 = 3/5 \\ B_2 & q_2 = 2/5 \end{bmatrix}$$

∴ value of the game

$$V = \frac{ad - bc}{(a+d) - (b+c)} = \frac{(5\times4) - (1\times3)}{(5+4) - (1+3)} = 17/5$$

∴ A won the game.

(2) Determine the optimum strategy and value of the game for given payoff

(a)

$$\begin{array}{ccc} & & P.B \\ & B_1 & B_2 \\ P.A \quad A_1 & 6 & -3 \\ A_2 & -3 & 0 \end{array}$$

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{rowmin} \\ A_1 & 6 & -3 & -3 \\ A_2 & -3 & 0 & -3 \\ \text{colmax} & 6 & 0 & \end{array}$$

minmax = min $\{6, 0\} = 6$

maxmin = max $\{-3, -3\} = -3$

minmax ≠ maxmin

$6 \neq -3$

∴ Saddle point does not exist

∴ Consider the probabilities P_1, P_2, q_1, q_2

$$P_1 = \frac{d - c}{(a+d) - (b+c)} = \frac{0 - (-3)}{(6+0) - (-3-3)} = \frac{3}{6+6} = \frac{\cancel{3}^{1}}{\cancel{12}_{4}} = \frac{1}{4}$$

$$P_2 = 1 - P_1 = 1 - \frac{3}{12} = \frac{9}{12} = \frac{3}{4}$$

$$q_1 = \frac{d - b}{(a+d) - (b+c)} = \frac{0 - (-3)}{(6+0) - (-3-3)} = \frac{3}{6+6} = \frac{3}{12} = \frac{1}{4}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{4} = \frac{3}{4}$$

∴ The mixed strategies for both the players.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 1/4 & P_2 = 3/4 \end{bmatrix} \quad S_B = \begin{bmatrix} B_1 & q_1 = 1/4 \\ B_2 & q_2 = 3/4 \end{bmatrix}$$

∴ value of the game

$$V = \frac{ad - bc}{(a+d) - (b+c)} = \frac{6(0) - (-3)(-3)}{(6+0) - (-3-3)} = \frac{-9}{12} = \frac{-3}{4}$$

∴ B won the game.

(b)

$$\begin{array}{ccc} & B_1 & B_2 \\ A_1 & \left[1 \right. & \left. -1/2 \right] \\ A_2 & \left[-1/2 \right. & \left. 0 \right] \end{array} \rightarrow P.B$$

P.A

	B_1	B_2	rowmin
A_1	1	$-1/2$	$-1/2$
A_2	$-1/2$	0	$-1/2$
colmax	1	0	

minmax = $\min\{1, 0\} = 1$

maxmin = $\max\{-1/2, -1/2\} = -1/2$

minmax $\neq$ maxmin

$1 \neq -1/2$

$\therefore$ Saddle point does not exist

$\therefore$ consider probabilities P_1, P_2, q_1, q_2

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{0-(-1/2)}{(1+0)-(-1/2-1/2)} = \frac{1/2}{1+1} = \frac{1}{4}$$

$$P_2 = 1 - P_1 = 1 - \frac{1}{4} = 3/4$$

$$q_1 = \frac{d-b}{(a+b)-(b+c)} = \frac{0-(-1/2)}{(1+0)-(-1/2-1/2)} = \frac{1/2}{1+1} = \frac{1}{4}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{4} = 3/4$$

$\therefore$ The mixed strategies for both the players

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 1/4 & P_2 = 3/4 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = 1/4 \\ B_2 & q_2 = 3/4 \end{bmatrix}$$

$\therefore$ Value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{0+\frac{1}{4}}{1+1} = \frac{1}{8}$$

$\therefore$ A won the game.

Dominance rule :-

The rules of dominance are used to reduce the size of payoff matrix.

(i) For player A, if each element in a row R_i or I $\leq$ each element in the row or R_j then delete row R_i.

(ii) For player B, if each element in the column $C_i \leq$ each element the column C_j then delete column C_j.

(iii) Some strategies also be dominated if it is inferior to on average or more than pure strategies.

① Find the value of the game of the given payoff

$$\begin{array}{cccc} & & \text{P.A} & \\ & B_1 & B_2 & B_3 \end{array}$$

$$\text{P.B} \quad \begin{array}{c} A_1 \\ A_2 \\ A_3 \end{array} \begin{bmatrix} -5 & 10 & 20 \\ 5 & -10 & -10 \\ 5 & -20 & -20 \end{bmatrix}$$

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \min \\ \hline A_1 & -5 & 10 & 20 & -5 \\ A_2 & 5 & -10 & -10 & -10 \\ A_3 & 5 & -20 & -20 & -20 \\ \hline \max & 5 & 10 & 20 & \end{array}$$

minmax $= \min\{5, 10, 20\} = 5$

maxmin $= \max\{-5, -10, -20\} = -5$

Here,

minmax $\neq$ maxmin

$\therefore$ No saddle point.

Apply dominance method.

Row which consists low value

Compare the rows A_2 & $A_3 \Rightarrow A_3 \leq A_2$, delete the row A_3

reduced matrix

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & -5 & 10 & 20 \\ A_2 & 5 & -10 & -10 \end{array}$$

highest value containing column.

Compare the columns B_2 & $B_3 \Rightarrow B_2 \leq B_3$, delete the column B_3

The reduced matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -5 & 10 \\ A_2 & 5 & -10 \end{array}$$

Consider the probilities P_1, P_2 & q_1, q_2

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-10-5}{(-5-10)-(10+5)} = \frac{-15}{-15-15} = \frac{-15}{-30} = 1/2$$

$$P_2 = 1 - P_1 = 1 - 1/2 = 1/2$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-10-10}{-15-15} = \frac{-20}{-30} = \frac{2}{3}$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{3} = 1/3$$

$\therefore$ The mixed strategy for both the players

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 1/2 & P_2 = 1/2 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = 2/3 \\ B_2 & q_2 = 1/3 \end{bmatrix}$$

$\therefore$ value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{50-50}{-15-15} = \frac{0}{-30} = 0$$

$\therefore$ It is a tie.

② Solve the given payoff

P.B

P.A

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 3 & 2 & 4 & 0 \\ A_2 & 3 & 4 & 2 & 4 \\ A_3 & 4 & 2 & 4 & 0 \\ A_4 & 0 & 4 & 0 & 4 \end{array}$$

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \min \\ \hline A_1 & 3 & 2 & 4 & 0 & 0 \\ A_2 & 3 & 4 & 2 & 4 & 2 \\ A_3 & 4 & 2 & 4 & 0 & 0 \\ A_4 & 0 & 4 & 0 & 4 & 0 \\ \hline \max & 4 & 4 & 4 & 4 & \end{array}$$

minmax $= \min\{4,4,4,4\} = 4$

maxmin $= \max\{0,2,0,0\} = 0$ [sic]

minmax $\neq$ maxmin

$\therefore$ No saddle point

Apply dominance method.

compare the rows A_1 & $A_3 \Rightarrow A_1 \le A_3$, delete A_1 row.

$$\begin{array}{c} \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \\ 0 & 4 & 0 & 4 \end{bmatrix} \end{array}$$

Compare the rows A_2 & $A_4 \Rightarrow A_4 \le A_2$, delete A_4 row.

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 3 & 4 & 2 & 4 \\ 4 & 2 & 4 & 0 \end{bmatrix} \end{array}$$

compare the colums B_1 & $B_3 \Rightarrow B_3 \le B_1$, delete B_1 column

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{ccc} B_2 & B_3 & B_4 \end{array} \\ \begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 0 \end{bmatrix} \end{array}$$

compare the columns B_2 & $B_4 \Rightarrow B_4 \le B_2$, delete B_2 column

$$\begin{array}{c} \\ A_2 \\ A_3 \end{array} \begin{array}{c} \begin{array}{cc} B_3 & B_4 \end{array} \\ \begin{bmatrix} 2 & 4 \\ 4 & 0 \end{bmatrix} \end{array}$$

consider the probabilities p_1, p_2, q_1, q_2

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{0-4}{(2+0)-(4+4)} = \frac{-4}{2-8} = \frac{-4}{-6} = \frac{2}{3}$$

$$p_2 = 1 - p_1 = 1 - \frac{2}{3} = \frac{1}{3}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{0-4}{-6} = \frac{-4}{-6} = \frac{2}{3}$$

$$q_2 = 1 - q_1 = 1 - \frac{2}{3} = 1/3$$

The mixed strategy for both the players

$$S_A = \begin{bmatrix} A_2 & A_3 \\ p_1 = 2/3 & p_2\ 1/3 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_3 & q_1 = 2/3 \\ B_4 & q_2 = 1/3 \end{bmatrix}$$

$\therefore$ value of the game

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{0-16}{-6} = \frac{8}{3}$$

$\therefore$ player A won the game.

③ Solve the following payoff $\begin{bmatrix} 1 & 7 & 3 & 4 \\ 5 & 6 & 4 & 5 \\ 7 & 2 & 0 & 3 \end{bmatrix}$.

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{min} \\ A_1 & 1 & 7 & 3 & 4 & 1 \\ A_2 & 5 & 6 & 4 & 5 & 4 \\ A_3 & 7 & 2 & 0 & 3 & 0 \\ \text{max} & 7 & 7 & 4 & 5 & \end{array}$$

minmax = min{7, 7, 4, 5} = 4

maxmin = max{1, 4, 0} = 4

minmax = maxmin

∴ Saddle point exist.

Apply dominance method.

compare the columns B_2 & B_3, $B_3 \le B_2$ then delete B_2 column

The reduced matrix is

$$\begin{array}{c|ccc} & B_1 & B_3 & B_4 \\ A_1 & 1 & 3 & 4 \\ A_2 & 5 & 4 & 5 \\ A_3 & 7 & 0 & 3 \end{array}$$

compare the rows A_1 & A_2, $A_1 \le A_2$ then delete A_1 row

$$\begin{array}{c|ccc} & B_1 & B_3 & B_4 \\ A_2 & 5 & 4 & 5 \\ A_3 & 7 & 0 & 3 \end{array}$$

compare the columns B_1 & B_3, $B_3 \le B_1$ then delete B_1 column

$$\begin{array}{c|cc} & B_3 & B_4 \\ A_2 & 4 & 5 \\ A_3 & 0 & 3 \end{array}$$

compare the rows A_2 & A_3, $A_3 \le A_2$ then delete A_3 row.

$$\begin{array}{c|cc} & B_3 & B_4 \\ A_2 & 4 & 5 \end{array}$$

compare the columns B_3 & B_4, $B_3 \le B_4$ then delete B_4 column

$$\begin{array}{c|c} & B_3 \\ A_2 & 4 \end{array}$$

∴ The pure strategy is $\{A_2, B_3\}$

value of the game = 4

④ Find the value of the game and optimal strategies for given payoff

ⓐ

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ A_1 & 7 & 6 & 2 & 7 \\ A_2 & 8 & 9 & 9 & 8 \\ A_3 & 7 & 5 & 8 & 5 \end{array}$$

ⓑ

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ A_1 & 3 & 6 & 1 \\ A_2 & 5 & 2 & 3 \\ A_3 & 4 & 2 & -5 \end{array}$$

(a)

$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \text{min} \\ \hline A_1 & 8 & 6 & 2 & 8 & 2 \\ A_2 & 8 & 9 & 4 & 8 & 4 \\ A_3 & 7 & 5 & 3 & 5 & 3 \\ \hline \text{max} & 8 & 9 & 4 & 8 & \end{array}$$

minmax = min{8, 9, 4, 8} = 4

maxmin = max{2, 4, 3} = 4

minmax = maxmin

∴ saddle point exist.

Apply dominance method.

compare the rows A_1 & A_2, $A_1 \le A_2$ then delete A_1 row.

$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_2 & 8 & 9 & 4 & 8 \\ A_3 & 7 & 5 & 3 & 5 \end{array}$$

compare the columns B_1 & B_4; $B_4 \le B_1$ then delete B_1 column

$$\begin{array}{c|ccc} & B_2 & B_3 & B_4 \\ \hline A_2 & 9 & 4 & 8 \\ A_3 & 5 & 3 & 5 \end{array}$$

compare the B_2 & B_4, $B_4 \le B_2$ then delete B_2 column

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_2 & 4 & 8 \\ A_3 & 3 & 5 \end{array}$$

compare the A_2 & A_3, $A_3 \le A_2$ then delete A_3 row

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_2 & 4 & 8 \end{array}$$

compare the columns B_3, B_4, $B_3 \le B_4$ then delete B_4 column

$$\begin{array}{c|c} & B_3 \\ \hline A_2 & 4 \end{array}$$

∴ The pure strategy is $\{A_2, B_3\}$

value of the game = 4

(b)

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{min} \\ \hline A_1 & 3 & 6 & 1 & 1 \\ A_2 & 5 & 2 & 3 & 2 \\ A_3 & 4 & 2 & -5 & -5 \\ \hline \text{max} & 5 & 6 & 3 & \end{array}$$

minmax = min{5, 6, 3} = 3

maxmin = max{1, 2, −5} = 2

minmax ≠ maxmin

∴ No saddle point

Apply dominance method.

compare the rows A_2 & A_3, $A_3 \le A_2$ then delete A_3 ~~column~~ row

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 3 & 6 & 1 \\ A_2 & 5 & 2 & 3 \end{array}$$

compare the columns B_1 & B_3, $B_3 \le B_1$ then delete B_1 column

$$\begin{array}{c|cc} & B_2 & B_3 \\ \hline A_1 & 6 & 1 \\ A_2 & 2 & 3 \end{array}$$

consider the probabilites p_1, p_2, q_1, q_2

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{3-2}{(6+3)-(1+2)} = \frac{1}{9-3} = \frac{1}{6} = 0.2$$

$$p_2 = 1 - p_1 = 1 - \frac{1}{6} = \frac{5}{6}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{3-1}{(6+3)-(1+2)} = \frac{2}{6} = \frac{1}{3}$$

$$q_2 = 1 - q_1 = 1 - \frac{1}{3} = \frac{2}{3}$$

The mixed strategy for both the players

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 = \frac{1}{6} & p_2 = \frac{5}{6} \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = \frac{1}{3} \\ B_2 & q_2 = \frac{2}{3} \end{bmatrix}$$

$\therefore$ value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{18-2}{6} = \frac{16}{6} = \frac{8}{3}$$

$\therefore$ player A won the game

1) solve the following 2×3 game graphically.

$$\begin{array}{ccc} & & P.B \\ & B_1 & B_2 & B_3 \\ P.A \begin{array}{c} A_1 \\ A_2 \end{array} & \left[\begin{array}{ccc} 1 & 3 & 11 \\ 18 & 5 & 2 \end{array}\right] \end{array}$$

$$\begin{array}{cccccc} & B_1 & B_2 & B_3 & & \text{min} \\ A_1 & \left[\begin{array}{c} 1 \\ \cancel{8}8 \end{array}\right. & \begin{array}{c} 3 \\ 5 \end{array} & \left.\begin{array}{c} 11 \\ 2 \end{array}\right] & & \begin{array}{c} 1 \\ 2 \end{array} \\ \text{max} & \cancel{8}8 & 5 & 11 & & \end{array}$$

minmax = min $\{\cancel{8}8, 5, 11\}$ = 5

maxmin = max $\{1, 2\}$ = 2

$\therefore$ minmax $\neq$ maxmin

$\therefore$ No saddle point exist.

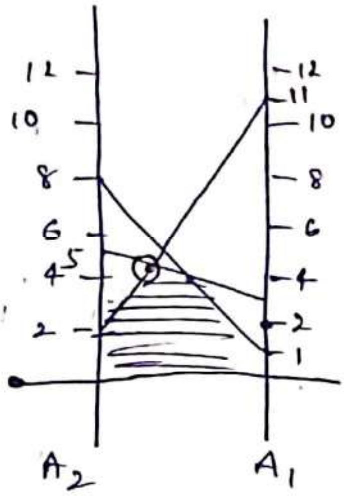
The expected payoff of player A

Pure strategy of B's	Expected payoff of A's.
B_1	$P_1 + 8P_2 = P_1 + 8(1-P_1) = 8 - 7P_1$
B_2	$3P_1 + 5P_2 = 3P_1 + 5(1-P_1) = 5 - 2P_1$
B_3	$11P_1 + 2P_2 = 11P_1 + 2(1-P_1) = 2 + 9P_1$

Now player A maximize his expected pay off we consider the highest point of intersection that the lines passing through the intersection.

$\therefore$ The reduced matrix is

$$\begin{array}{c c} & \begin{array}{cc} B_2 & B_3 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} 3 & 11 \\ 5 & 2 \end{bmatrix} \end{array}$$



consider the probobilities p_1, p_2 & q_1, q_2

$$p_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2-5}{(2+3)-(11+5)} = \frac{-3}{-11} = 3/11$$

$$p_2 = 1-p_1 = 1-\frac{3}{11} = \frac{9}{11}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{2-11}{(2+3)-(11+5)} = \frac{-9}{-11} = \frac{9}{11}$$

$$q_2 = 1-q_1 = \frac{2}{11}$$

∴ The mixed strategy for both the players ;

$$S_A = \begin{bmatrix} A_1 & A_2 \\ p_1 = 3/11 & p_2 = 9/11 \end{bmatrix}, \quad S_B = \begin{bmatrix} B_1 & q_1 = 9/11 \\ B_2 & q_2 = 2/11 \end{bmatrix}$$

value of the game $= V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{6-55}{-11} = \frac{-49}{-11} = \frac{49}{11}$

∴ player A won the game.

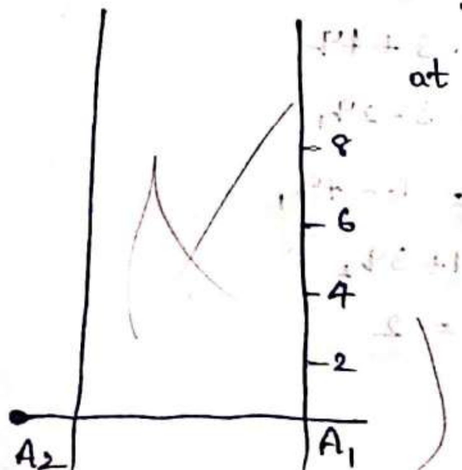
② Solve the following game graphically
$$\begin{array}{c c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -6 & 0 & 6 & -3/2 \\ 7 & -3 & -8 & 2 \end{bmatrix} \end{array}$$

$$\begin{array}{c c c} & \begin{array}{cccc} B_1 & B_2 & B_3 & B_4 \end{array} & \text{min} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -6 & 0 & 6 & -3/2 \\ 7 & -3 & -8 & 2 \end{bmatrix} & \begin{array}{c} -6 \\ -8 \end{array} \\ \text{max} & \begin{array}{cccc} 7 & 0 & 6 & 2 \end{array} & \end{array}$$

~~no~~ minmax = 0
maxmin = −6
∴ minmax ≠ maxmin
∴ No saddle point exist.

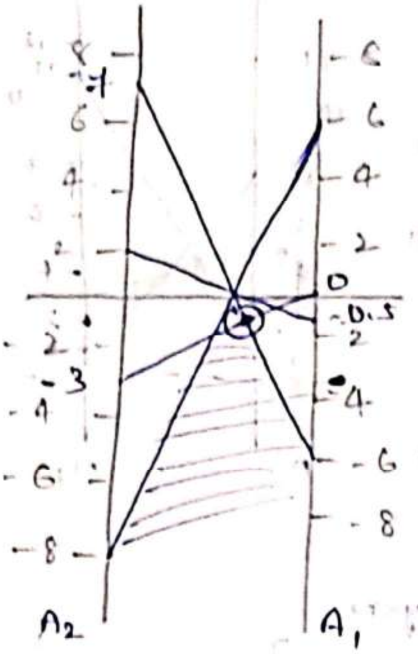
The expected payoff of player A

The pure strategy of B's	Expected payoff of A's.
B_1	$-6P_1 + 7P_2 = -6P_1 + 7(1-P_1) = 7 - 13P_1$
B_2	$0P_1 + (-3)(1-P_1) = -3 + 3P_1$
B_3	$6P_1 - 8P_2 = 6P_1 - 8(1-P_1) = -8 + 14P_1$
B_4	$-\frac{3}{2}P_1 + 2P_2 = -\frac{3}{2}P_1 + 2(1-P_1) = 2 + \frac{1}{2}P_1$



The highest on lower envelope appear at the intersection of lines. The reduced game is

$$\begin{array}{c c} & \begin{array}{cc} B_1 & B \end{array} \\ \begin{array}{c} A_1 \\ A_2 \end{array} & \begin{bmatrix} -6 & 0 \\ 7 & -3 \end{bmatrix} \end{array}$$



The reduced matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -6 & 0 \\ A_2 & 7 & -3 \end{array}$$

consider the probabilities P_1, P_2 & q_1, q_2

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-3-7}{-9-7} = \frac{-10}{-16} = \frac{5}{8}$$

$$P_2 = 1 - P_1 = 1 - \frac{5}{8} = \frac{3}{8}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-3-0}{-9-7} = \frac{-3}{-16} = \frac{3}{16}$$

$$q_2 = 1 - q_1 = 1 - \frac{3}{16} = \frac{13}{16}$$

The mixed strategy of both the players.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = \frac{5}{8} & P_2 = \frac{3}{8} \end{bmatrix}; \quad S_B = \begin{bmatrix} B_1 & q_1 = \frac{3}{16} \\ B_2 & q_2 = \frac{13}{16} \end{bmatrix}$$

∴ The value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{18-0}{-9-7} = \frac{18}{-16} = -\frac{9}{8}$$

∴ B won the game

③ Solve the following game graphically

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & -3 \\ A_2 & 3 & 5 \\ A_3 & -1 & 6 \\ A_4 & 4 & 1 \\ A_5 & 2 & 2 \\ A_6 & -5 & 0 \end{array}$$

$$\begin{array}{c|cc|c} & B_1 & B_2 & \\ \hline A_1 & 1 & -3 & -3 \\ A_2 & 3 & 5 & 3 \\ A_3 & -1 & 6 & -1 \\ A_4 & 4 & 1 & 1 \\ A_5 & 2 & 2 & 2 \\ A_6 & -5 & 0 & 0 \\ \hline & 4 & 6 & \end{array}$$

minmax = 4

maxmin = 3

minmax ≠ maxmin

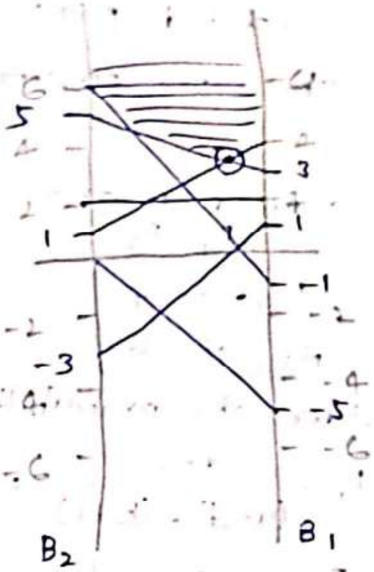
∴ No saddle point exist.

Apply graphical method.

∴ The expected payoff of player B

Pure strategy A's	Expected payoff of B's
A_1	$q_1 - 3q_2 = q_1 - 3(1-q_1) = -3 + 4q_1$
A_2	$3q_1 + 5q_2 = 3q_1 + 5(1-q_1) = 5 - 2q_1$
A_3	$-1q_1 + 6q_2 = -q_1 + 6(1-q_1) = 6 - 7q_1$
A_4	$4q_1 + q_2 = 4q_1 + 1 - q_1 = 1 + 3q_1$
A_5	$2q_1 + 2q_2 = 2q_1 + 2(1-q_1) = 2$
A_6	$-5q_1 + 0 = -5q_1$

the expected point is least point of intersection i.e., the lines passing through the intersection.



The reduced matrix is

$$\begin{array}{c|c} & \begin{array}{cc} B_1 & B_2 \end{array} \\ \begin{array}{c} A_2 \\ A_4 \end{array} & \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \end{array}$$

$$P_1 = \frac{1-4}{(1+3)-(4+5)} = \frac{3}{5}$$

$$P_2 = 1 - P_1 = 2/5$$

$$q_1 = \frac{1-5}{(1+3)-(4+5)} = 4/5$$

$$q_2 = 1 - q_1 = 1/5$$

mixed strategy for both players.

$$S_A = \begin{bmatrix} A_2 & A_4 \\ P_1 = 3/5 & P_2 = 2/5 \end{bmatrix} \quad S_B = \begin{bmatrix} B_1 & q_1 = 4/5 \\ B_2 & q_2 = 1/5 \end{bmatrix}$$

value of the game

$$V = \frac{ad - bc}{(a+d)-(b+c)} = \frac{3-20}{4-9} = \frac{-17}{-5} = 17/5$$

∴ Player A won the game

(4) solve the game graphically

$$\begin{array}{c|c|c} & \begin{array}{cc} B_1 & B_2 \end{array} & \text{min} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} & \begin{pmatrix} -6 & 7 \\ 4 & -5 \\ -1 & -2 \\ -2 & 5 \\ 7 & 6 \end{pmatrix} & \begin{array}{c} -6 \\ -5 \\ -2 \\ -2 \\ 6 \end{array} \\ \text{max} & \begin{array}{cc} 7 & 7 \end{array} & \end{array}$$

-6 7
4 -5
-1 -2
-2 5
7 6

minmax = 7

maxmin = 6

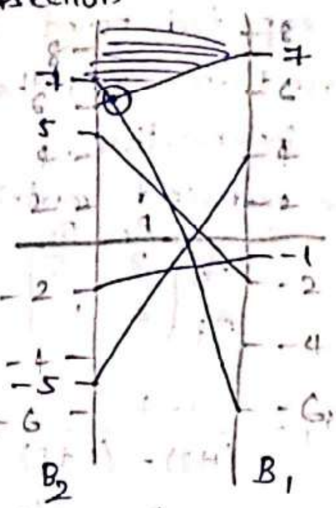
minmax ≠ maxmin

∴ No saddle point exist.

Apply graphical method

Pure strategy A's	Expected payoff of B's
A_1	$-6q_1 + 7q_2 = -6q_1 + 7(1-q_1) = 7 - 13q_1$
A_2	$4q_1 - 5q_2 = 4q_1 - 5(1-q_1) = -5 - 9q_1$
A_3	$-1q_1 - 2q_2 = -q_1 - 2(1-q_1) = -2 + q_1$
A_4	$-2q_1 + 5q_2 = -2q_1 + 5(1-q_1) = 5 - 7q_1$
A_5	$7q_1 + 6q_2 = 7q_1 + 6(1-q_1) = 6 + q_1$

The expected point is least point of intersection i.e, the lines passing through intersection.



The reduced matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & -6 & 7 \\ A_5 & 7 & 6 \end{array}$$

Consider the probabilities P_1, P_2, q_1, q_2

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{6-7}{0-14} = \frac{-1}{-14} = \frac{1}{14}$$

$$P_2 = 1 - P_1 = \frac{13}{14}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{6-7}{0-14} = \frac{-1}{-14} = \frac{1}{14}$$

$$q_2 = 1 - q_1 = \frac{13}{14}$$

The mixed strategy for both players,

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 1/14 & P_2 = 13/14 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = 1/14 \\ B_2 & q_2 = 13/14 \end{bmatrix}$$

The value of game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{-36-49}{-14} = \frac{85}{14}$$

$\therefore$ player A won the game.

⑤ solve the following game

$$\begin{array}{c|ccc} & B_1 & B_2 & B_3 \\ \hline A_1 & 3 & -2 & 4 \\ A_2 & -1 & 4 & 2 \\ A_3 & 2 & 2 & 6 \end{array}$$

$$\begin{array}{c|ccc|c} & B_1 & B_2 & B_3 & \text{min} \\ \hline A_1 & 3 & -2 & 4 & -2 \\ A_2 & -1 & 4 & 2 & -1 \\ A_3 & 2 & 2 & 6 & 2 \\ \hline \text{max} & 3 & 4 & 6 & \end{array}$$

minmax = 2

maxmin = 3

minmax $\neq$ maxmin

$\therefore$ No saddle point exist

Apply dominance method.

compare the columns B_1 & B_3 ; $B_1 \leq B_3$ then delete B_3.

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 3 & -2 \\ A_2 & -1 & 4 \\ A_3 & 2 & 2 \end{array}$$

compare the row : It cannot be reduced into further.

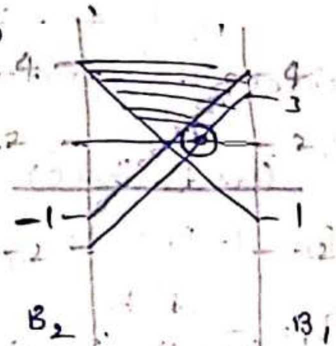
Apply graphical method.

A_1 $\quad 3q_1 - 2q_2 = 3q_1 - 2(1-q_1) = -2 + 5q_1$

A_2 $\quad -1q_1 + 4q_2 = -q_1 + 4(1-q_1) = 4 - 5q_1$

A_3 $\quad 2q_1 + 2q_2 = 2q_1 + 2(1-q_1) = 2$

The expected point is least point of intersection i.e., lines passing through intersection points.



The reduced matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 3 & -2 \\ A_3 & 2 & 2 \end{array}$$

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{2-2}{5-0} = 0$$

$P_2 = 1 - P_1 = 1$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{2+2}{5-0} = \frac{4}{5}$$

$q_2 = 1 - q_1 = 1 - 4/5 = 1/5$.

The mixed strategy for both the game

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 0 & P_2 = 1 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = 4/5 \\ B_2 & q_2 = 1/5 \end{bmatrix}$$

The value of game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{6+4}{5-0} = \frac{10}{5} = 2$$

$\therefore$ player A won the game.

ⓐ
$$\begin{array}{c|cccc} & B_1 & B_2 & B_3 & B_4 \\ \hline A_1 & 1 & 3 & -3 & 1 \\ A_2 & 2 & 5 & 4 & -6 \end{array}$$

ⓑ
$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_1 & 1 & 2 \\ A_2 & 5 & 4 \\ A_3 & -7 & 9 \\ A_4 & -4 & -3 \\ A_5 & 2 & 1 \end{array}$$

ⓐ
$$\begin{array}{c|cccc|c} & B_1 & B_2 & B_3 & B_4 & \min \\ \hline A_1 & 1 & 3 & -3 & 1 & -3 \\ A_2 & 2 & 5 & 4 & -6 & -6 \\ \hline \max & 2 & 5 & 4 & 1 & \end{array}$$

minmax = 1

maxmin = -3

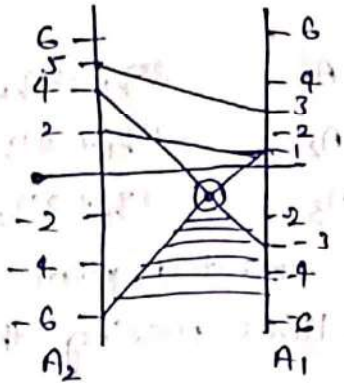
$\therefore$ minmax $\neq$ maxmin

$\therefore$ No saddle point exist

B's	Expected payoff of A's
B_1	$P_1 + 2P_2 = P_1 + 2(1-P_1) = 2 - P_1$
B_2	$3P_1 + 5P_2 = 3P_1 + 5(1-P_1) = -2P_1 + 5$
B_3	$-3P_1 + 4P_2 = -3P_1 + 4(1-P_1) = 4 - 7P_1$
B_4	$P_1 - 6P_2 = P_1 - 6(1-P_1) = -6 + 7P_1$

The reduced matrix is

$$\begin{array}{c|cc} & B_3 & B_4 \\ \hline A_1 & -3 & 1 \\ A_2 & 4 & -6 \end{array}$$



consider the probabilities P_1, P_2 & q_1, q_2

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{-6-4}{-9-5} = \frac{-10}{-14} = \frac{5}{7}$$

$$P_2 = 1-P_1 = 1-\frac{5}{7} = \frac{2}{7}$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{-6-1}{-9-5} = \frac{-7}{-14} = \frac{1}{2}$$

$$q_2 = 1-q_1 = 1-\frac{1}{2} = \frac{1}{2}$$

The mixed strategy for both the players

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = 5/7 & P_2 = 2/7 \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = 1/2 \\ B_2 & q_2 = 1/2 \end{bmatrix}$$

The value of the game

$$v = \frac{ad-bc}{(a+d)-(b+c)} = \frac{18-4}{-9-5} = \frac{14}{-14} = -1$$

$\therefore$ B won the game.

(b)

$$\begin{array}{c|cc|c} & B_1 & B_2 & \text{min} \\ \hline A_1 & 1 & 2 & 1 \\ A_2 & 5 & 4 & 4 \\ A_3 & -7 & 9 & -7 \\ A_4 & -4 & -3 & -4 \\ A_5 & 2 & 1 & 1 \\ \hline \text{max} & 5 & 9 & \end{array}$$

minmax = 5

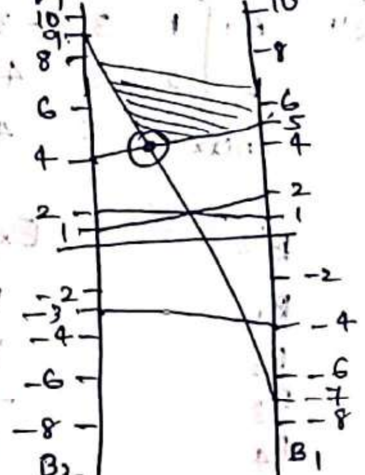
maxmin = 4

$\therefore$ minmax $\neq$ maxmin

$\therefore$ No saddle point exist

Apply graphical method.

A's	Expected payoff of B's
A_1	$q_1 + 2q_2 = q_1 + 2(1-q_1) = 2-q_1$
A_2	$5q_1 + 4q_2 = 5q_1 + 4(1-q_1) = 4+q_1$
A_3	$-7q_1 + 9q_2 = -7q_1 + 9(1-q_1) = 9-16q_1$
A_4	$-4q_1 - 3q_2 = -4q_1 - 3(1-q_1) = -3-q_1$
A_5	$2q_1 + q_2 = 2q_1 + 1 - q_1 = 1+q_1$



The reduced matrix is

$$\begin{array}{c|cc} & B_1 & B_2 \\ \hline A_2 & 5 & 4 \\ A_3 & -7 & 9 \end{array}$$

consider the probabilities P_1, P_2 & q_1, q_2

$$P_1 = \frac{d-c}{(a+d)-(b+c)} = \frac{9+7}{14+3} = \frac{16}{17}$$

$$P_2 = 1 - P_1 = 1 - \frac{16}{17} = 1/17$$

$$q_1 = \frac{d-b}{(a+d)-(b+c)} = \frac{9-4}{14+3} = \frac{5}{17}$$

$$q_2 = 1 - q_1 = 1 - \frac{5}{17} = \frac{12}{17}$$

The mixed strategy for both the players.

$$S_A = \begin{bmatrix} A_1 & A_2 \\ P_1 = \frac{16}{17} & P_2 = \frac{1}{17} \end{bmatrix} \qquad S_B = \begin{bmatrix} B_1 & q_1 = \frac{5}{17} \\ B_2 & q_2 = \frac{12}{17} \end{bmatrix}$$

The value of the game

$$V = \frac{ad-bc}{(a+d)-(b+c)} = \frac{45+28}{14+3} = \cancel{\frac{61}{17}} = \frac{73}{17}$$

$\therefore$ player A won the game.